

Boundary Integral Equations in Acoustic Scattering Survey

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Abstract

These notes are a self-contained survey of boundary integral equation methods for acoustic scattering, following the course structure of Barnett's *Numerical analysis for PDEs and wave scattering* [1] and drawing throughout on Kress [7, 8] and Colton [2, 3]. Very little here is new, our aim is to assemble the standard theory in one place and to make explicit the dependencies between its parts. We begin with the functional analysis of compact integral operators, developing Riesz and Fredholm theory in dual systems, since these supply the existence and uniqueness statements that everything downstream rests on. We then treat the Laplace equation, where layer potentials and their jump conditions are introduced in their simplest setting, and we study the spectrum of the Laplacian itself. Turning to the Helmholtz equation, we reformulate the exterior sound-soft and sound-hard scattering problems as boundary integral equations and show that the natural formulations fail at interior resonances. The combined field integral equation repairs this for all real wave numbers, and its spurious modes are located through the impedance eigenvalue problem of the Laplacian. We close with a brief account of how these equations are discretized in practice, covering kernel approximation, the Nyström method, and the quadrature corrections that singular kernels demand.

Contributions

These notes are the product of a shared reading course ran by Bryce Parry and Wietse Vaes, under the watchful gaze and guidance of Heather Wilber. Responsibility for the material is divided as follows. Bryce Parry prepared, presented, and wrote Sections 1.1-1.3, Sections 3.1-3.3, and Chapter 4. Wietse Vaes prepared, presented, and wrote Chapter 2 and Section 3.4. Emily Zhang presented the Fredholm theory of Section 1.4. Levent Batakci presented a particular solution form for the exterior Helmholtz problem on the circle. The typesetting and editing of this document were carried out by Bryce Parry and Wietse Vaes.

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Chapter 1

Integral Operators & Fredholm Theory

Boundary integral equation methods rest on a single observation: an elliptic boundary value problem posed on a domain can be recast as an integral equation posed only on its boundary, and the operator that comes out is a compact perturbation of the identity. This chapter is the theoretical groundwork for exploiting that structure, and it stands apart from any particular PDE so that the later chapters can lean on it without pausing. We follow [8] throughout. We first collect the properties of bounded and compact operators, and show that integral operators with continuous or weakly singular kernels are compact, which is exactly why the layer potentials of the coming chapters are. We then develop the Riesz theory of $I - A$, whose payoff is that injectivity alone gives us existence and continuous dependence on the data, so for a second-kind formulation it is enough to prove that the homogeneous problem has only the trivial solution. Lastly, we turn to Fredholm theory, which tells us what happens when injectivity fails. This is the case we meet at interior resonances, and it is what lets us pin down exactly the wave numbers at which the standard boundary integral formulations break down.

1.1 Bounded and Compact Operators

We will work almost exclusively with compact operators. The properties and results we will present will allow us to prove very nice existence and uniqueness statements for equations involving such operators.

1.1.1 Linear Bounded Operators

We start with linear bounded operators.

Definition 1.1.1 (Linear and Bounded Operator). Let X, Y be normed spaces, and define an operator $A : X \rightarrow Y$. We say A is *linear* if

$$A(\alpha\varphi + \beta\psi) = \alpha A\varphi + \beta A\psi$$

for all $\alpha, \beta \in \mathbb{C}$ and $\varphi, \psi \in X$.

The operator A is *bounded* if there exists a positive constant C such that

$$\|A\varphi\| \leq C\|\varphi\|$$

We say that every C this is true for is a bound on the operator.

We can interpret an operator as bounded if the image under the operation does not grow more than linearly. Alternatively, a linear operator $A : X \rightarrow Y$ is bounded if and only if A maps bounded sets in X to bounded sets in Y .

Theorem 1.1.2. *Let X and Y be normed spaces. A linear operator $A : X \rightarrow Y$ is bounded if and only if*

$$\|A\| := \sup_{\|\varphi\|=1} \|A\varphi\| < \infty$$

The number $\|A\|$ is the smallest bound for A and is referred to as the norm of the operator.

Proof. $\Rightarrow$) This is true by the definition of a bounded operator with $\|\varphi\| < 1 \iff$) Take $\varphi \in X$

$$\|A\varphi\| = \left\| A \frac{\|\varphi\|}{\|\varphi\|} \varphi \right\| = \|\varphi\| \left\| A \frac{\varphi}{\|\varphi\|} \right\| \leq \sup_{\|\psi\|=1} \|A\psi\| \|\varphi\|$$

□

Remark 1.1.3. We note that any linear operator from a finite dimensional normed space into a normed space is bounded. This relies on the result that all norms are equivalent on a finite dimensional norm space.

It is true that linear combinations of bounded linear operators are bounded linear operators.

Theorem 1.1.4. *Let X, Y, Z be normed spaces and define the bounded linear operators $A : X \rightarrow Y$ and $B : Y \rightarrow Z$. Then $BA : X \rightarrow Z$ is a bounded linear operator with norm $\|(BA)\varphi\| \leq \|A\| \|B\|$*

Proof. This follows from linearity properties of A and B and Cauchy-Schwartz. □

This lets us build new bounded linear operators from existing bounded linear operators. It also enables us to analyze sets of linear operators like sets of vectors using the tools of linear spaces.

Now we present the notion of convergence that we will need to analyze these spaces of operators.

Definition 1.1.5 (Cauchy Sequence). A *Cauchy sequence* is a sequence (φ_n) such that

$$\lim_{n,m \rightarrow \infty} \|\varphi_n - \varphi_m\| = 0$$

That is elements in the sequence keep getting closer together.

Definition 1.1.6 (Complete Set). A subset $U \subseteq X$ is *complete* if every Cauchy sequence of elements in U converges to an element in U .

Definition 1.1.7 (Banach Space). A complete normed space is called a *Banach Space*.

Theorem 1.1.8. *Let X, Y be normed spaces. The linear space $L(X, Y)$ of bounded linear operators $A : X \rightarrow Y$ is a normed space with norm*

$$\|A\| := \sup_{\|\varphi\|=1} \|A\varphi\| < \infty$$

Further, if Y is a Banach space then $L(X, Y)$ is a Banach space.

Proof. The spirit of the proof is to evoke the completeness and norm properties of Y to show $L(X, Y)$ is a normed space since the operator norm is really looking at the norm of elements in Y with the norm defined on Y . $\square$

With this result we have a way to measure how close operators are in a space. Naturally, we can consider sequences of operators.

Definition 1.1.9 (Operator Convergence). A sequence of operators $(A_n) \subset L(X, Y)$ is *norm convergent* to an operator $A \in L(X, Y)$ if $A_n \rightarrow A$ as $n \rightarrow \infty$ in the $L(X, Y)$ norm.

A sequence $(A_n) \subset L(X, Y)$ is *point-wise convergent* to an operator A if $A_n\varphi \rightarrow A\varphi$ as $n \rightarrow \infty$ for all $\varphi \in X$.

Note that norm convergence is the stronger convergence. Indeed, norm convergence implies point-wise convergence but the inverse is not true.

Definition 1.1.10 (Isomorphic Operator). A bijective bounded linear operator $A : X \rightarrow Y$ is an *isomorphism* if the inverse operator $A^{-1} : Y \rightarrow X$ is bounded.

We say two normed spaces are *isomorphic* if there exists an isomorphism between them.

Note that isomorphic spaces have the same convergent sequences.

We are ready to introduce our next major player in our functional analysis lineup: the dual space.

Definition 1.1.11 (Dual space). Let X be a normed space. The *dual space* of X is the set of bounded linear operators and is denoted $X^* := L(X, \mathbb{C})$.

The dual space is a Banach space with norm

$$\|F\| := \sup_{\|\varphi\|=1} |F(\varphi)|, \quad F \in X^*, \varphi \in X$$

Notice that elements of X^* are bounded linear functionals taking an element of X to a complex scalar. It is also tempting to ask “what functionals are in the dual space”? The Hahn-Banach theorem ensures that any dual space has more than the trivial zero functional in it and is worth studying.

Theorem 1.1.12 (Hahn-Banach theorem). *Let X be a normed space. Take $U \subset X$ and $F : U \rightarrow \mathbb{C}$. Then there exists bounded linear functional $G : X \rightarrow \mathbb{C}$ with the properties*

1. $G(\varphi) = F(\varphi)$ for $\varphi \in U$
2. $\|G\| = \|F\|$

So the Hahn-Banach theorem allows us to extend any linear functional defined on a subset to the whole space such that the extension agrees on the subset, and the norm does not grow.

1.1.2 Compact Operators

Next we can move to the realm of compact operators.

Definition 1.1.13 (Relatively compact set). A subset of a normed space is *relatively compact* (alternatively pre-compact) if its closure is compact. Or if every sequence in the subset has a convergent subsequence, but the limit need not be in the subset.

The Arzelà-Ascoli theorem gives an alternative way to determine if a subset is relatively compact or not.

Theorem 1.1.14 (Arzelà-Ascoli). Define $C(G)$ as the set of continuous functions defined on a compact set $G \subset \mathbb{C}^m$. A set $U \subset C(G)$ is relatively compact if and only if it is bounded and equicontinuous. That is there exists a C such that $|\varphi(x)| \leq C \forall x \in G$ and $\forall \varphi \in U$ (this is bounded) and $\forall \epsilon > 0, \exists \delta > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon \forall x, y \in G$ with $|x - y| < \delta$ (this is equicontinuous).

Equicontinuous is upgrading continuity to a whole set of functions such that the $\epsilon - \delta$ pair at a point holds for all of the functions in the set $C(G)$.

Definition 1.1.15 (Compact operator). A linear operator $A : X \rightarrow Y$ is *compact* if it maps each bounded set in X to a relatively compact set in Y .

Again, we have a theorem that gives us an alternative lens to view a compact operator

Theorem 1.1.16. $A : X \rightarrow Y$ is compact if and only if for each bounded sequence $(\varphi_n) \subset X$ the sequence $(A\varphi_n) \in Y$ contains a convergent subsequence.

Proof. $\implies$) A set is relatively compact $\implies$ each sequence in the set contains a convergent subsequence. $\impliedby$) Each sequence in the set contains a convergent subsequence $\implies$ the set is relatively compact $\implies$ A maps a compact set to a relatively compact set. $\square$

Now we list some important properties of compact operators.

Observation 1.1.17. Compact linear operators are bounded.

Observation 1.1.18. Linear combinations of compact linear operators are compact.

Observation 1.1.19. Let X, Y, Z be normed spaces, and $A : X \rightarrow Y, B : Y \rightarrow Z$ be bounded linear operators. The operator $BA : X \rightarrow Z$ is compact if A or B is compact.

Observation 1.1.20. Bounded linear operators with finite dimensional range are compact operators.

Observation 1.1.21. Let (A_n) be a sequence of compact linear operators from normed space X to a Banach space Y that is norm convergent to a linear operator $A : X \rightarrow Y$. Then A is compact.

This last property is useful when we have an operator A that is not so nice to work with, but we can approximate it with more well behaved operators. Then we can say something about the compactness of A .

All in all, we should view compact operators as keeping elements close together from one space to another. These compact operators are very nice to work with, so it is good to know when

an operator is or is not compact. The Riesz lemma gives us a tool to answer the negative question.

Lemma 1.1.22 (Riesz). *Let X be a normed space and $U \subset X$ a closed subspace. Take $\alpha \in (0, 1)$. Then there exists a $\psi \in X$ with $\|\psi\| = 1$ such that $\|\psi - \varphi\| \geq \alpha$ for all $\varphi \in U$.*

An immediate consequence of this lemma is that a subset being closed is not enough for it to be relatively compact. Thus an operator mapping to a closed set is not necessarily compact. It also allows us to prove the following theorem about the compactness of the identity operator.

Theorem 1.1.23. *The identity operator $I : X \rightarrow X$ is compact if and only if X has finite dimension.*

Proof. $\Leftarrow$) I is a finite dimensional operator that is bounded and linear. Thus by Observation 1.1.20 I is compact.

$\Rightarrow$) We prove this direction with contradiction. Assume I is compact and not finite dimensional. Choose $\varphi_1 \in X$ with $\|\varphi_1\| = 1$. Denote $U_1 := \text{span}(\varphi_1)$. U_1 is finite dimensional and thus a closed subspace. Apply Lemma 1.1.22, so there exists $\varphi_2 \in X$ with $\|\varphi_2\| = 1$ and $\|\varphi_2 - \varphi_1\| \geq 1/2$. Now define $U_2 := \text{span}(U_1, \varphi_2)$. This is finite dimensional and closed, so apply Lemma 1.1.22 again. We can repeatedly apply the lemma to generate a sequence (φ_n) with $\|\varphi_n\| = 1$ and $\|\varphi_n - \varphi_m\| \geq 1/2$ for $n \neq m$. So no subsequence converges! Thus a contradiction on the compact assumption of I . $\square$

This is an important result as we will see the identity operator plays a central role in the so called integral equations of the second kind we will encounter later.

1.2 Integral Operators

A mainstay in these notes are integral operators. These will allow us to define the namesake boundary integral equations which are well used in the solutions of elliptic PDEs. They are used in problems arising in acoustic and electromagnetic scattering, viscoelastic fluid flow, and fracture mechanics, to name a few.

In this section we will define an integral operator and show that they can be compact operators.

1.2.1 Introduction

First, we need to state some preliminaries.

Definition 1.2.1 (Characteristic Function). The characteristic function χ_A is one such that for a set A

$$\chi_A(a) = \begin{cases} 1, & a \in A \\ 0, & a \notin A \end{cases}$$

Definition 1.2.2 (Jordan Measurable). A set $G \subset \mathbb{R}^m$ is *Jordan measurable* if $\chi_G(x)$ is Riemann integrable.

We define the *Jordan measure* $|G|$ as the integral of χ_G .

For any Jordan measurable G , $\overline{G}$ is Jordan measurable with measure $|\overline{G}| = |\overline{G}|$. Additionally, ∂G is Jordan measurable with measure $|\partial G| = 0$. Colloquially, we can say that the Jordan measure does not “see” the boundary.

Also, we have that if G is compact and Jordan measurable, then every continuous function defined on G is Riemann integrable.

Definition 1.2.3. Take $G \in \mathbb{R}^m$ to be a nonempty, compact, Jordan measurable set that is the closure of its interior. Take $K : G \times G \rightarrow \mathbb{C}$ to be a continuous function.

The linear operator $A : C(G) \rightarrow C(G)$ defined by

$$[A\varphi](x) := \int_G K(x, y)\varphi(y) dy, \quad x \in G$$

is an *integral operator* with a continuous kernel K .

Theorem 1.2.4. An integral operator with continuous kernel K as defined in Definition 1.2.3 is a bounded linear operator.

It has norm

$$\|A\|_\infty = \max_{x \in G} \int |K(x, y)| dy$$

Integral operators can be used to form operator equations, which we can attempt to solve for the unknown. This is the idea of boundary integral equations we explore in these notes. A differential operator can be expressed as an integral operator with a kernel specific to the PDE. Then an equation is formed with respect to the known boundary data, and we can solve the equation this way.

With this goal in mind, we define operator and integral equations.

Definition 1.2.5 (Operator Equations). Let $A : X \rightarrow Y$ be an operator. An *operator equation of the first kind* is $A\varphi = f$ where $\varphi \in X$ is unknown and $f \in Y$ is known.

An *operator equation of the second kind* is $(I - A)\varphi = f$.

An integral equation is an operator equation where A is an integral operator. It turns out that equations of the second kind are much more well behaved mathematically and numerically. For this reason we will primarily concern ourselves with integral equations of the second kind.

We can study these integral operators with the tools we have previously developed

Theorem 1.2.6. Integral operators with continuous kernels are compact linear operators on $C(G)$.

Proof. Let A be an integral operator with continuous kernels K , for all $\varphi \in C([a, b])$ with $\|\varphi\|_\infty \leq 1$ and all $x \in [a, b]$, we have that

$$|(A\varphi)(x)| \leq \int_a^b |K(x, y)\varphi(y)| dy \leq \int_a^b dy \max_{x, y \in [a, b]} |K(x, y)| \|\varphi\|_\infty \leq (b - a) \max_{x, y \in [a, b]} |K(x, y)|,$$

i.e., the set $U := \{A\varphi \mid \varphi \in C([a, b]), \|\varphi\|_\infty \leq 1\} \subset C([a, b])$ is bounded. Since K is continuous, K is uniformly continuous on the compact square $[a, b] \times [a, b]$, for every $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$|K(x, z) - K(y, z)| < \frac{\epsilon}{b - a},$$

for all $x, y \in [a, b]$ with $|x - y| < \delta$. Then

$$|(A\phi)(x) - (A\phi)(y)| = \left| \int_a^b [K(x, z) - K(y, z)]\phi(z) \, dz \right| \leq \epsilon,$$

for all $x, y \in [a, b]$ with $|x - y| < \delta$ and all $\phi \in C([a, b])$ with $\|\phi\|_\infty \leq 1$, i.e., U is equicontinuous. Hence, by the Arzelà-Ascoli Theorem 1.1.14, A is compact. $\square$

Note that we presented the proof for continuous functions on G , but the result can be upgraded to functions $\phi \in L^2(G)$. We omit this proof. However, as we will see, many integral operators we will encounter in the wild will not have continuous kernels. We will prepare for these situations with the following definitions and theorems.

Definition 1.2.7 (Weakly Singular Kernel). A kernel $K : G \times G \rightarrow \mathbb{C}$ is *weakly singular* if it is defined and continuous for all $x, y \in G \subseteq \mathbb{R}^m$ with $x \neq y$ and there exist $M > 0$ and $\alpha \in (0, m]$ such that

$$|K(x, y)| \leq \frac{M}{|x - y|^{m-\alpha}}$$

In other words, the singularity of the kernel is of order less than the dimension of the space. Thus, it blows up, but not too fast that it is not integrable. We would be remiss to not mention that many kernels of practical importance have a singularity with order equal to or greater than the dimension. These are called *hypersingular* kernels. We will not treat these kernels in these notes, but they are of great practical importance.

Theorem 1.2.8. *Integral operators with weakly singular kernels are compact linear operators on $C(G)$.*

Proof. The idea of the proof is to estimate the integral operator with weakly singular kernel with a sequence of integral operators with continuous kernels. Thus they are compact operators. Showing that the sequence is norm convergent to the operator with weakly singular kernel then tells us that it is compact by Observation 1.1.21.

We do so by defining the sequence of kernels

$$K_n(x, y) = \begin{cases} h(n|x - y|)K(x, y), & x \neq y \\ 0, & x = y \end{cases}$$

where

$$h(t) = \begin{cases} 0, & 0 \leq t \leq 1/2 \\ 2t - 1, & 1/2 \leq t \leq 1 \\ 1, & t > 1 \end{cases}$$

These K_n cut off the singularity and replace them with 0. But as $n \rightarrow \infty$ we converge to the original, weakly-singular kernel K . The K_n are continuous so if we consider the sequence of integral operators with kernel K_n we have integral operators with continuous kernels. Thus they are compact by Theorem 1.2.6. Then as $n \rightarrow \infty$, Observation 1.1.21 tells us the integral operator with kernel K is compact. $\square$

1.2.2 Volterra Integral Operators

We briefly present the case of Volterra integral equations, which can be thought of as Fredholm equations where the kernel is zero valued for $y > x$.

Definition 1.2.9. Integral equations of the form

$$\int_a^x K(x, y)\varphi(y) dy = f(x), \quad x \in [a, b]$$

and

$$\varphi(x) - \int_a^x K(x, y)\varphi(y) dy = f(x), \quad x \in [a, b]$$

are *Volterra integral equations* of the first and second kind, respectively.

Note that for each right hand side of the integral equations, $f \in C[a, b]$ the Volterra integral equation of the second kind with a continuous kernel has a unique solution $\varphi \in C[a, b]$. Such a general statement cannot be made about the first kind equations, and in general working with a second kind, as said for Fredholm equations, is preferable. In many cases an equation of the first kind can be turned into a second kind equation by taking a derivative with respect to the non-integrating variable. We will see this in action later in the notes when we visit the Laplace and subsequently Helmholtz equations.

1.3 Riesz Theory

As mentioned a few times now, we want to solve operator equations of the form $(I - A)\varphi = f$ where A is an integral operator. We have just shown that an integral operator with continuous or weakly singular kernels are compact. Now we develop the next tool to hang on our belt: Riesz Theory. In this section, define $L := I - A$, i.e. a perturbation of the identity. We present three Riesz theorems.

Theorem 1.3.1 (First Riesz Theorem). *The nullspace of the operator L , denoted $\mathbf{N}(L) := \{\varphi \in X : L\varphi = 0\}$ is a finite dimensional subspace.*

Theorem 1.3.2 (Second Riesz Theorem). *The range of the operator L denoted $L(X) := \{L\varphi : \varphi \in X\}$ is a closed linear subspace.*

Theorem 1.3.3 (Third Riesz Theorem). *There exists a unique $r \geq 0$ called the Riesz number of the operator A such that*

$$\underbrace{\{0\} = \mathbf{N}(L^0) \subsetneq \mathbf{N}(L^1) \subsetneq \cdots \subsetneq \mathbf{N}(L^r) = \mathbf{N}(L^{r+1}) = \cdots}_{\text{nullspace grows } \rightarrow}$$

and

$$\underbrace{X = L^0(X) \supsetneq L^1(X) \supsetneq \cdots \supsetneq L^r(X) = L^{r+1}(X) = \cdots}_{\text{range shrinks } \rightarrow}$$

and we have the direct sum

$$X = \mathbf{N}(L^r) \oplus L^r(X)$$

This tells us that for this operator L , we keep taking powers of it, and we are guaranteed to at some point we split the whole space into the nullspace and range of L .

We say that an operator is of *finite ascent* if two successive iterated operators have the same nullspace. An operator is of *finite descent* if two successive iterated operators have the same range. In simpler words, an operator is of finite ascent if the nullspace levels out under iterations, and of finite descent if the range levels out. Note that if A is a compact linear operator, then $I - A$ is of finite ascent and descent.

Theorem 1.3.4. *Let X be a normed space and define a compact linear operator $A : X \rightarrow X$. Then $I - A$ is injective if and only if $I - A$ is surjective.*

Proof. Let $I - A$ be injective, then $\mathbf{N}(L^0) = \{0\}$ which means $r = 0$. Let $I - A$ be surjective, then $X = L^0(X)$ which means $r = 0$. Thus injective and surjective are both equivalent to $r = 0$. $\square$

Theorem 1.3.5. *Let X be a normed space and define a compact linear operator $A : X \rightarrow X$. If $I - A$ is injective (therefore bijective) then $(I - A)^{-1} : X \rightarrow X$ is bounded and $I - A$ is an isomorphism.*

Proof. We will prove by contradiction. Assume $(I - A)^{-1}$ is not bounded. This means we can find a sequence $(f_n) \subset X$ with $\|f_n\| = 1$ such that $\|L^{-1}f_n\| \geq n$ for all $n \in \mathbb{N}$. Construct the following two sequences

$$g_n := \frac{f_n}{\|L^{-1}f_n\|}, \quad \varphi_n := \frac{L^{-1}f_n}{\|L^{-1}f_n\|}, \quad n \in \mathbb{N}$$

Clearly, $g_n \rightarrow 0$ as $n \rightarrow \infty$ and $\|\varphi_n\| = 1$ for all n . A is compact so we get a subsequence of (φ_n) that is convergent in X . Now consider

$$\varphi_n - A\varphi_n = (I - A)\varphi_n = (I - A) \frac{(I - A)^{-1}f_n}{\|L^{-1}f_n\|} = \frac{f_n}{\|L^{-1}f_n\|} = g_n$$

So we have

$$\varphi_n - A\varphi_n = g_n$$

Now take $n \rightarrow \infty$ and we see

$$(I - A)\varphi = 0 \implies \varphi \in \mathbf{N}(I - A)$$

This would mean $\|\varphi\| = 0$. Contradiction on $\|\varphi_n\| = 1$ for all n ! $\square$

With this last theorem, we have a result on the inverse of the operator $I - A$. This puts us in the position to say something about the solution of equations with this operator!

Corollary 1.3.6. *Let X be a normed space and $A : X \rightarrow X$ be a compact linear operator. If the homogeneous equation*

$$(I - A)\varphi = 0$$

has only the trivial solution $\varphi = 0$, then for all $f \in X$, the inhomogeneous equation

$$(I - A)\varphi = f$$

has a unique solution that depends continuously on the data f .

Proof. If we only have the trivial solution $\varphi = 0$ then $\mathbf{N}(I - A) = \{0\}$ which means $I - A$ is injective thus bijective and thus an isomorphism. $\square$

Corollary 1.3.7. *Let X be a normed space and $A : X \rightarrow X$ be a compact linear operator. If the homogeneous equation*

$$(I - A)\varphi = 0$$

has a nontrivial solution, then it only has a finite number $m \in \mathbb{N}$ of linearly independent solutions $\varphi_1, \dots, \varphi_m$ and the inhomogeneous equation

$$(I - A)\varphi = f$$

is either unsolvable or has a general solution

$$\varphi = \tilde{\varphi} + \sum_{k=1}^m \alpha_k \varphi_k$$

where $\alpha_k \in \mathbb{C}$ and $\tilde{\varphi}$ is a particular solution of the homogeneous equation.

In the Riesz theory, we have looked at operators of the form $I - A$ where I is the identity. We can generalize the results by replacing I by any bounded operator with a bounded inverse.

Corollary 1.3.8. *The previous three results (Theorem 1.3.5 and Corollaries 1.3.6 and 1.3.7) hold when I is replaced by S where $S : X \rightarrow Y$ is a bounded linear operator with a bounded inverse $S^{-1} : Y \rightarrow X$.*

Proof.

$$(S - A)\varphi = (I - S^{-1}A)\varphi$$

and $S^{-1}A$ is compact by Observation 1.1.19 since A is compact and S^{-1} is bounded. $\square$

Since we have seen that integral operators with continuous and weakly singular kernels are compact, we can apply the Riesz theory to $I - A$ where A is an integral operator! The theory then allows us to replace the problem of establishing uniqueness of a solution to $(I - A)\varphi = f$ with the problem of showing $(I - A)\varphi = 0$ only has $\varphi = 0$ as a solution. The latter is (generally) an easier problem.

1.3.1 Riesz Theory as Spectral Theory

We can frame the results of Riesz theory through the spectral analysis of compact operators. For this, we need more definitions, though hopefully some will be familiar to the reader.

Definition 1.3.9. Let X be a normed space and $A : X \rightarrow X$ a bounded linear operator.

- $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there exists $\varphi \in X$, $\varphi \neq 0$, such that $A\varphi = \lambda\varphi$.
- $\varphi \in X$, $\varphi \neq 0$ such that $A\varphi = \lambda\varphi$ is called an *eigenelement* of A .
- $\lambda \in \mathbb{C}$ is called a *regular value* of A if $(\lambda I - A)^{-1} : X \rightarrow X$ exists and is bounded.
- The set of all regular values is the *resolvent set* $\rho(A)$.
- The *resolvent* of A is the operator $R(\lambda; A) := (\lambda I - A)^{-1}$.
- The complement of the resolvent set is the *spectrum* $\sigma(A)$.

- The *spectral radius* of A is

$$r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|$$

Theorem 1.3.10. *Let X be an infinite dimensional norm space and $A : X \rightarrow X$ a compact linear operator. Then $\lambda = 0$ belongs to the spectrum $\sigma(A)$ and $\sigma(A) \setminus \{0\}$ consists of at most a countable set of eigenvalues with no point accumulation except, possibly, at $\lambda = 0$.*

1.4 Fredholm Theory

The compactness of integral operators is the backbone of boundary integral equations, in particular, the study of equation of the form $(I + K)\varphi = f$. In many ways, compact operators behave like finite-dimensional operators, despite the fact that they are acting on infinite-dimensional spaces. Fredholm theory characterizes the solvability of equations involving the identity plus a compact operator and establishes the backbone between invertibility and the presence of a finite-dimensional null-space or co-kernel. The treatment here follows [8].

1.4.1 Dual Systems

Riesz theory already tells us that $(I - A)$ is injective if and only if it is surjective, but it says nothing about what happens when injectivity fails. In finite dimensions we know the answer: $Mx = b$ is solvable precisely when b is orthogonal to the null-space of M^T . To carry that statement into infinite dimensions we need a notion of transpose, and for that we need a pairing between two spaces. The pairing is supplied by a bilinear form.

Definition 1.4.1 (Bilinear form). Let X, Y be linear spaces. A mapping

$$\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{C}$$

is called a bilinear form if

$$\begin{aligned} \langle \alpha_1 \varphi_1 + \alpha_2 \varphi_2, \psi \rangle &= \alpha_1 \langle \varphi_1, \psi \rangle + \alpha_2 \langle \varphi_2, \psi \rangle, \\ \langle \varphi, \beta_1 \psi_1 + \beta_2 \psi_2 \rangle &= \beta_1 \langle \varphi, \psi_1 \rangle + \beta_2 \langle \varphi, \psi_2 \rangle \end{aligned}$$

for all $\varphi_1, \varphi_2, \varphi \in X$, $\psi_1, \psi_2, \psi \in Y$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. The bilinear form is called non-degenerate if for every $\varphi \in X$ with $\varphi \neq 0$ there exists $\psi \in Y$ such that $\langle \varphi, \psi \rangle \neq 0$, and for every $\psi \in Y$ with $\psi \neq 0$ there exists $\varphi \in X$ such that $\langle \varphi, \psi \rangle \neq 0$.

Non-degeneracy is the condition that makes the pairing informative: no non-zero element of either space is invisible to the other. Without it, a statement such as “ f is orthogonal to the null-space of the adjoint” would carry no content, since a large subspace could be orthogonal to everything. A pair of spaces carrying such a form is exactly the structure we work in.

Definition 1.4.2 (Dual system). Two normed spaces X and Y equipped with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{C}$ are called a dual system and denoted by $\langle X, Y \rangle$.

Two dual systems will be used throughout. The first is available for free on any normed space, at the cost of carrying the dual space around in the second slot.

Theorem 1.4.3. *Each normed space X together with its dual space X^* forms the canonical dual system $\langle X, X^* \rangle$ with bilinear form*

$$\langle \varphi, F \rangle := F(\varphi), \quad \varphi \in X, F \in X^*.$$

The proof relies on the Hahn-Banach Theorem 1.1.12, which supplies, for every non-zero φ , a functional that does not annihilate it [8].

The second dual system is the one we actually want for boundary integral equations. Rather than pairing $C(G)$ with the space of measures that constitutes its dual, we pair $C(G)$ with itself through integration. This keeps both slots concrete and, as we will see, means the adjoint of an integral operator is again an integral operator on the same space.

Theorem 1.4.4. *Let $G \subset \mathbb{R}^m$ be a nonempty compact and Jordan measurable set that coincides with the closure of its interior. Then $\langle C(G), C(G) \rangle$ is a dual system with the bilinear form*

$$\langle \varphi, \psi \rangle := \int_G \varphi(x) \psi(x) dx, \quad \varphi, \psi \in C(G).$$

Proof. Bilinearity is immediate from linearity of the integral, so only non-degeneracy needs an argument. Let $\varphi \in C(G)$ with $\varphi \not\equiv 0$, say $\varphi(x_0) \neq 0$. Since G is the closure of its interior, x_0 can be taken in the interior of G , and by continuity there is a ball $B_\epsilon(x_0) \subset G$ on which $|\varphi| > \frac{1}{2}|\varphi(x_0)|$ and on which φ does not change argument. Choose $\psi \in C(G)$ non-negative, supported in $B_\epsilon(x_0)$ and equal to $\overline{\varphi}/|\varphi|$ times a bump there. Then $\langle \varphi, \psi \rangle \neq 0$. Jordan measurability guarantees the integral is defined in the Riemann sense for all of $C(G)$. The roles of the two slots are symmetric, so the same argument applies in reverse. $\square$

The hypotheses on G are exactly what this proof consumes: without G being the closure of its interior, a non-zero φ could be supported on a lower-dimensional sliver of G and be invisible to every pairing. With a pairing in hand we can transpose operators.

Definition 1.4.5 (Adjoint). Let $\langle X_1, Y_1 \rangle$ and $\langle X_2, Y_2 \rangle$ be two dual systems. Then two operators $A : X_1 \rightarrow X_2$, $B : Y_2 \rightarrow Y_1$ are called adjoint (w.r.t. these dual systems) if

$$\langle A\varphi, \psi \rangle = \langle \varphi, B\psi \rangle$$

for all $\varphi \in X_1, \psi \in Y_2$.

Note that the definition assumes neither linearity nor boundedness of either operator. Both come for free, and the adjoint is pinned down completely by the pairing.

Theorem 1.4.6. *Let $\langle X_1, Y_1 \rangle$ and $\langle X_2, Y_2 \rangle$ be two dual systems. If an operator $A : X_1 \rightarrow X_2$ has an adjoint $B : Y_2 \rightarrow Y_1$, then B is uniquely determined and A and B are linear.*

Proof. Suppose B_1 and B_2 are both adjoint to A . Then $\langle \varphi, (B_1 - B_2)\psi \rangle = 0$ for every $\varphi \in X_1$ and every $\psi \in Y_2$, and non-degeneracy in the second slot forces $(B_1 - B_2)\psi = 0$. For linearity of A , take $\varphi_1, \varphi_2 \in X_1$ and scalars α_1, α_2 . For every $\psi \in Y_2$,

$$\langle A(\alpha_1\varphi_1 + \alpha_2\varphi_2), \psi \rangle = \langle \alpha_1\varphi_1 + \alpha_2\varphi_2, B\psi \rangle = \langle \alpha_1A\varphi_1 + \alpha_2A\varphi_2, \psi \rangle,$$

using bilinearity of the form and the adjoint relation twice. Non-degeneracy in the first slot gives $A(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1A\varphi_1 + \alpha_2A\varphi_2$. The argument for B is identical. $\square$

Uniqueness is settled, but existence is a genuine question. In the canonical dual system it is always answered affirmatively.

Theorem 1.4.7. *Let $A : X \rightarrow Y$ be a bounded linear operator mapping a normed space X into a normed space Y . Then the adjoint operator $A^* : Y^* \rightarrow X^*$ w.r.t. the canonical dual systems X, X^* and $\langle Y, Y^* \rangle$ exists. It is given by*

$$A^*F := FA, \quad F \in Y^*,$$

and is called the dual operator of A . It is bounded with norm $\|A\| = \|A^*\|$.

In the dual system $\langle C(G), C(G) \rangle$ the adjoint is far more explicit, and this is the statement that makes Fredholm theory usable in practice. Transposing an integral operator amounts to swapping the arguments of its kernel.

Theorem 1.4.8. *Let K be a continuous or weakly singular kernel. Then in the dual system $\langle C(G), C(G) \rangle$ the (compact) integral operators defined by*

$$(A\varphi)(x) := \int_G K(x, y)\varphi(y) \, dy, \quad x \in G,$$

and

$$(B\psi)(x) := \int_G K(y, x)\psi(y) \, dy, \quad x \in G,$$

are adjoint.

Proof. For $\varphi, \psi \in C(G)$,

$$\langle A\varphi, \psi \rangle = \int_G \left(\int_G K(x, y)\varphi(y) \, dy \right) \psi(x) \, dx = \int_G \varphi(y) \left(\int_G K(x, y)\psi(x) \, dx \right) dy = \langle \varphi, B\psi \rangle,$$

where the interchange is Fubini's theorem. For a continuous kernel the double integral is over a compact set with a continuous integrand, so the hypothesis is trivially met. For a weakly singular kernel, $|K(x, y)| \leq C|x - y|^{\alpha-m}$ with $\alpha > 0$, and the majorant $|\varphi(y)\psi(x)||x - y|^{\alpha-m}$ is integrable over $G \times G$, so Fubini again applies. $\square$

This is the exact analogue of $M \mapsto M^T$ for matrices, and it is worth keeping in mind that the adjoint here is a transpose and not a conjugate transpose. Note also that B is the operator whose kernel is $K(y, x)$, which in the layer potential setting is precisely the relation between the double-layer operator $\mathcal{D}$ and its adjoint $\mathcal{D}'$. The conjugated version of all of this requires a sesquilinear pairing instead.

Definition 1.4.9 (Sesquilinear form). Let X, Y be linear spaces, A mapping

$$(\cdot, \cdot) : X \times Y \rightarrow \mathbb{C}$$

is called a sesquilinear form if

$$\begin{aligned} (\alpha_1\varphi_1 + \alpha_2\varphi_2, \psi) &= \alpha_1(\varphi_1, \psi) + \alpha_2(\varphi_2, \psi) \\ (\varphi, \beta_1\psi_1 + \beta_2\psi_2) &= \overline{\beta_1}(\varphi, \psi_1) + \overline{\beta_2}(\varphi, \psi_2) \end{aligned}$$

for all $\varphi_1, \varphi_2, \varphi \in X$, $\psi_1, \psi_2, \psi \in Y$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$.

The sesquilinear form is related to the bilinear form

$$(\varphi, \psi) = \langle, \varphi, \psi^* \rangle,$$

except the sesquilinear form is no longer holomorphic. The most common sesquilinear form on G for $\varphi, \psi \in C(G)$ is given by

$$(\varphi, \psi) = \int_G \varphi(x) \overline{\psi(x)} dx,$$

with an associated norm

$$\|\varphi\| = \sqrt{(\varphi, \varphi)}$$

The price of the conjugation is that antilinearity in the second slot breaks holomorphy in ψ , so the sesquilinear setting is the wrong one whenever we want to track dependence on a parameter analytically, such as the wave number k . Its advantage is that it comes with a norm, and with it the geometry of Hilbert space. The Riesz representation theorem then makes the canonical dual system of Theorem 1.4.3 collapse onto the space itself.

Theorem 1.4.10. *Let X be a Hilbert space. Then for each bounded linear functional $F : X \rightarrow \mathbb{C}$ there exists a unique element $f \in X$ s.t.*

$$F(\varphi) = (\varphi, f)$$

for all $\varphi \in X$. The norms of the element f and the linear function F coincide, i.e.,

$$\|f\| = \|F\|.$$

Identifying X^* with X in this way turns the dual operator into an operator on the original spaces, which is the familiar Hilbert space adjoint.

Theorem 1.4.11. *Let X and Y be Hilbert spaces, and let $A : X \rightarrow Y$ be a bounded linear operator. Then there exists a uniquely determined linear operator $A^* : Y \rightarrow X$ with the property*

$$(A\varphi, \psi) = (\varphi, A^*\psi)$$

for all $\varphi \in X$ and $\psi \in Y$, i.e., A and A^* are adjoint w.r.t. the dual systems (X, X) and (Y, Y) generated by the scalar products on X and Y . The operator A^* is bounded and $\|A^*\| = \|A\|$.

Compactness also transfers across the adjoint [8], which we will need since both Fredholm theorems below require A and B to be compact, not just one of them.

Theorem 1.4.12. *Let X and Y be Hilbert spaces and let $A : X \rightarrow Y$ be a compact linear operator. Then the adjoint operator $A^* : Y \rightarrow X$ is also compact.*

There is one loose end before we move on to solvability. Everything built so far, compactness of weakly singular integral operators and the Riesz theory, lives in $C(G)$ with the supremum norm. The scalar product above induces the L^2 norm instead, and on $C(G)$ the two are not equivalent. Passing between them is not a formality we can wave through, and the following theorem of Lax [8] is the tool that does it. It converts sup-norm boundedness into L^2 boundedness for free, provided an adjoint relation is available, which by Theorem 1.4.8 is always the case for our layer operators.

Theorem 1.4.13 (Lax). *Let X and Y be normed spaces, both of which have a scalar product $(\cdot, \cdot)$, and assume that there exists a positive constant c such that*

$$|(\varphi, \psi)| \leq c \|\varphi\| \|\psi\|$$

for all $\varphi, \psi \in X$. Let U be a subspace of X and let $A : X \rightarrow Y$ and $B : Y \rightarrow X$ be bounded linear operators satisfyin

$$(A\varphi, \psi) = (\varphi, B\psi)$$

for all $\varphi \in U$ and $\psi \in Y$. Then $A : U \rightarrow Y$ is bounded with respect to the norms $\|\cdot\|_s$ induces by the scalar products and

$$\|A\|_s^2 \leq \|A\| \|B\|$$

The practical consequence is that an operator bounded on $C(G)$ with a transposed-kernel partner is bounded in the L^2 norm, and since $C(G)$ is dense in $L^2(G)$ it extends uniquely to a bounded operator on $L^2(G)$. This is how the single- and double-layer operators are moved off $C(\partial D)$ and onto $L^2(\partial D)$, which is where the hypersingular operator and the Dirichlet-to-Neumann map have to be analysed [8, 3].

1.4.2 Fredholm Alternative

We now have everything needed to state the solvability theory for $(I - A)\varphi = f$. Riesz theory already gives us that $\mathcal{N}(I - A)$ is finite-dimensional (Theorem 1.3.1) and that $(I - A)(X)$ is closed (Theorem 1.3.2). What Fredholm theory adds is that the null-space of the adjoint has the *same* dimension, and that it characterizes the range exactly. The bookkeeping in both proofs rests on the existence of a biorthogonal system, which is where non-degeneracy of the pairing is finally spent.

Lemma 1.4.14. *Let $\langle X, Y \rangle$ be a dual system. Then to every set of linearly independent elements $\varphi_1, \dots, \varphi_n \in X$ there exists a set $\psi_1, \dots, \psi_j \in Y$ such that*

$$\langle \varphi_i, \psi_j \rangle = \delta_{ij}, \quad i, k = 1, \dots, n,$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise. The same statement holds with the roles of X and Y interchanged.

The lemma is a dual basis statement, proved by induction on n [8], and it is what allows the two null-spaces to be compared dimension by dimension.

Theorem 1.4.15 (First Fredholm Theorem). *Let $\langle X, Y \rangle$ be a dual system and $A : X \rightarrow X$, $B : Y \rightarrow Y$ be compact adjoint operators. Then the nullspaces of the operators $I - A$ and $I - B$ have the same finite dimension.*

Finiteness of each dimension separately is the first Riesz Theorem 1.3.1; the content here is the equality of the two. The equality is not a soft statement and fails badly without compactness [8]. The unilateral shift on ℓ^2 is injective while its adjoint has a one-dimensional null-space, so no such balance holds for general bounded operators.

Knowing that the two null-spaces have matching size, the remaining question is which right-hand sides are attainable. The second theorem answers this with the orthogonality condition we set out to reach.

Theorem 1.4.16 (Second Fredholm Theorem). *Let $\langle X, Y \rangle$ be a dual system and $A : X \rightarrow X$, $B : Y \rightarrow Y$ be compact adjoint operators. Then*

$$(I - A)(X) = \{f \in X \mid \langle f, \psi \rangle = 0, \psi \in \mathcal{N}(I - B)\}$$

and

$$(I - B)(Y) = \{g \in Y \mid \langle \varphi, g \rangle = 0, \varphi \in \mathcal{N}(I - A)\}$$

Proof of the inclusion $\subseteq$. Let $f \in (I - A)(X)$, so $f = (I - A)\varphi$ for some $\varphi \in X$, and let $\psi \in \mathcal{N}(I - B)$. Since A and B are adjoint, so are $I - A$ and $I - B$, and therefore

$$\langle f, \psi \rangle = \langle (I - A)\varphi, \psi \rangle = \langle \varphi, (I - B)\psi \rangle = 0.$$

The second identity follows by the same computation with the roles of the spaces exchanged. $\square$

The reverse inclusion is the substantial half, and it is where closedness of the range from the second Riesz Theorem 1.3.2 and the biorthogonal system of Theorem 1.4.14 are needed; the full argument is given in [8]. Combining the two Fredholm theorems with the Riesz dichotomy gives the statement that is actually used in practice.

Theorem 1.4.17 (Fredholm Alternative). *Let $A : X \rightarrow X$, $B : Y \rightarrow Y$ be compact adjoint operators in a dual system $\langle X, Y \rangle$. Then either $I - A$ and $I - B$ are bijective or $I - A$ and $I - B$ have nontrivial nullspaces with finite dimension*

$$\dim \mathcal{N}(I - A) = \dim \mathcal{N}(I - B) \in \mathbb{N}$$

and the ranges are given by

$$(I - A)(X) = \{f \in X \mid \langle f, \psi \rangle = 0, \psi \in \mathcal{N}(I - B)\}$$

and

$$(I - B)(Y) = \{g \in Y \mid \langle \varphi, g \rangle = 0, \varphi \in \mathcal{N}(I - A)\}$$

The first case is the Riesz statement that injectivity implies surjectivity, now known to hold for $I - A$ and $I - B$ simultaneously by Theorem 1.4.15. The second case is Theorem 1.4.16. This is the infinite-dimensional analogue of the statement that a square matrix is either invertible or has a non-trivial kernel, and it is the reason second-kind formulations are so attractive: uniqueness implies existence, so for a boundary integral equation of the second kind it suffices to prove that the homogeneous problem has only the trivial solution. Specializing to the integral operators of Theorem 1.4.8 in the dual system of Theorem 1.4.4 gives the form in which we will actually invoke it.

Corollary 1.4.18. *Let $G \subset \mathbb{R}^m$ be as in Theorem 1.4.4 and let K be a continuous or weakly singular kernel. Then either the homogeneous integral equations*

$$\varphi(x) - \int_G K(x, y)\varphi(y) \, dy = 0, \quad x \in G,$$

and

$$\psi(x) - \int_G K(y, x)\psi(y) \, dy = 0, \quad x \in G,$$

only have the trivial solutions $\varphi = 0$ and $\psi = 0$ and the inhomogeneous integral equations

$$\varphi(x) - \int_G K(x, y) \varphi(y) \, dy = f(x), \quad x \in G,$$

and

$$\psi(x) - \int_G K(y, x) \psi(y) \, dy = g(x), \quad x \in G,$$

have a unique solution $\varphi \in C(G)$ and $\psi(G)$ for each $f \in C(G)$ and $g \in C(G)$, respectively, or the homogeneous integral equations have the same finite number $m \in \mathbb{N}$ of linearly independent solution and the inhomogeneous integral equations are solvable if and only if the right hand sides satisfy

$$\langle f, \psi \rangle = 0$$

for all solutions ψ of the homogeneous adjoint equation and

$$\langle \varphi, g \rangle = 0$$

for all solutions φ of the homogeneous equation, respectively.

No proof is needed, the corollary is Theorem 1.4.17 read in the dual system $\langle C(G), C(G) \rangle$, with Theorem 1.4.8 supplying the adjoint pair and compactness coming from the weak singularity of the kernel. This is the tool we use for the Laplace and Helmholtz layer potential formulations, where the adjoint equation is the one obtained by swapping source and target in the kernel, and the solvability conditions above become concrete compatibility conditions on the boundary data.

It is worth closing the loop with the spectral reading of Riesz theory. Applying the Fredholm Alternative to $I - \lambda^{-1}A$ for $\lambda \neq 0$ shows that λ is an eigenvalue of A if and only if it is an eigenvalue of the adjoint B , and by Theorem 1.4.15 the two eigenspaces have the same dimension. Combined with the fact that the non-zero spectrum of a compact operator consists of at most countably many eigenvalues accumulating only at the origin, this says $\sigma(A) \setminus \{0\} = \sigma(B) \setminus \{0\}$ with matching multiplicities. In the layer potential setting this is the statement that $\mathcal{D}$ and $\mathcal{D}'$ share a spectrum, which is why the interior Dirichlet and Neumann resonances appear symmetrically in the exterior sound-soft and sound-hard formulations [3].

Chapter 2

Laplace equation & Layer Potentials

The Laplace equation is perhaps the first elliptic PDE most people encounter. It is ubiquitous in the real world and for good reason! It describes physical states of harmony which can be seen using mathematical theorems on Harmonic functions. In real life, it can model steady state solutions of the heat equation, soap bubble films and electric voltage with no free electric charges. For us, the Laplace equation is a stepping stone to the Helmholtz equation. We discuss classical results on harmonic functions, to then prove the uniqueness of solutions to boundary value problems using the Laplace operator. Afterwards we discuss layer potentials in the Laplace equation setting to prove existence, and we note where the resulting second-kind formulations are well conditioned. These layer potentials are conceptually grounded using the electric voltage context, which is useful for the Helmholtz equation in our future. We then note what the smoothness assumptions on the boundary are actually buying us, and what changes at corners. Lastly, we study the spectrum of the Laplacian itself. This may look like a detour, but the interior eigenvalues are precisely the wave numbers at which the Helmholtz boundary integral formulations fail.

2.1 Harmonic Functions

Definition 2.1.1 (Harmonic functions). A twice differentiable real-valued function u , defined on domain $D \subset \mathbb{R}^m$ is called harmonic if it satisfies Laplace's equation

$$\Delta u = 0, \quad x \in D$$

We say that Δ is the Laplacian, and let $\vec{n}$ denotes the outward pointing unit normal vector on a curve, we write the normal derivative as $\frac{\partial u}{\partial \vec{n}}(x) = \vec{n}(x) \cdot \nabla u(x)$. Most of the theory of harmonic functions utilize Green's theorems [2, 4]. The first and second Green's theorems are practically a reformulation of Gauss' divergence formula.

Theorem 2.1.2 (Green's First Theorem). *Let D be a bounded domain of class C^1 , $\vec{n}$ denotes the outward pointing unit normal vector on ∂D . For $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$, we have*

$$\int_D u \Delta v + \nabla u \cdot \nabla v \, d\Omega = \int_{\partial D} u \frac{\partial v}{\partial \vec{n}} \, d\Gamma.$$

Theorem 2.1.3 (Green's Second Theorem). *Let D be a bounded domain of class C^1 , $\vec{n}$ denotes the outward pointing unit normal vector on ∂D . For $u, v \in C^2(\bar{D})$, we have*

$$\int_D u \Delta v - v \Delta u \, d\Omega = \int_{\partial D} u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}} \, d\Gamma$$

The second Green's theorem applied to an harmonic functions (with $v \equiv 1$), directly leads us to the corollary that harmonic functions observe net flux zero.

Corollary 2.1.4. *Let $u \in C^2(D)$ be harmonic in D , then*

$$\int_{\partial D} \frac{\partial u}{\partial \vec{n}} \, d\Gamma = 0$$

To study harmonic functions we first find the Green's function to the Laplacian, or the fundamental solution to the Laplace equation [4].

Theorem 2.1.5 (The Fundamental Solution). *The function*

$$\varphi(x, y) := \begin{cases} \frac{1}{2\pi} \ln \left(\frac{1}{|x - y|} \right), & d = 2 \\ \frac{\Gamma\left(\frac{d-1}{2}\right)}{(d-2)2\pi^{(d-1)/2}} \frac{1}{|x - y|^{d-2}}, & d \geq 3 \end{cases} \quad (2.1)$$

defined $\forall x \neq y$ in $\mathbb{R}^m$ is called the fundamental solution of Laplace's equation. For fixed $y \in \mathbb{R}^d$ it is harmonic in $\mathbb{R}^m \setminus \{y\}$

Proof. As the two-dimensional setting is our focus, we only prove the case for $d = 2$. The difference for higher dimensions will be noted. The fundamental solution to Laplace's equation is given by

$$-\Delta \varphi = \delta(x - y), \quad x \in \mathbb{R}^2, \quad (2.2)$$

with δ the dirac delta distribution. Since the Laplacian is rotation invariant, The fundamental solution observes radial symmetry, i.e., we can go to polar coordinates and use the ansatz

$$\varphi(x, y) = \psi(r), \quad \text{with } r = |x - y|,$$

and have

$$\Delta \varphi = \psi'' + \frac{d-1}{r} \psi' = 0.$$

Solving for ψ' gives us $\psi' = cr^{-(d-1)}$ such that for $d = 2$,

$$\psi = c \ln(r) + b.$$

In settings that have dimension higher than 2, it is typically argues that constant b is set to zero for the purpose of decay at zero with infinity. In the two-dimensional setting, the fundamental solution does not decay, but has logarithmic growth at infinity. The convention is to also set $b = 0$ here due to physical relevance and the fact that for the purpose of BIE, this constant would integrate away. To find the constant c , we integrate (2.2) over a ball centered at y with ϵ radius, $\mathbb{B}_\epsilon(y) := \{x \in \mathbb{R}^m \mid |x - y| < \epsilon\}$, $\mathbb{B}_\epsilon$ in short,

$$\int_{\mathbb{B}_\epsilon} \Delta \varphi \, d\Omega = \int_{\mathbb{B}_\epsilon} -\delta(x - y) \, d\Omega.$$

By definition of the dirac delta distribution and Green's first Theorem 2.1.2, we have

$$\int_{\mathbb{B}_\epsilon} \Delta \varphi \, d\Omega = \int_{\partial \mathbb{B}_\epsilon} \vec{n} \cdot \nabla \varphi \, d\Gamma = -1.$$

On the surface $\partial \mathbb{B}_\epsilon$, we have

$$\vec{n} = \frac{x - y}{\epsilon} \quad \text{and} \quad \nabla \varphi = c \frac{x - y}{|x - y|^2} = \frac{c}{\epsilon^2} (x - y),$$

thus,

$$\int_{\partial \mathbb{B}_\epsilon} \vec{n} \cdot \nabla \varphi \, d\Gamma = c \int_{\partial \mathbb{B}_\epsilon} \frac{|x - y|^2}{\epsilon^3} \, d\Gamma = \frac{c}{\epsilon} \int_{\partial \mathbb{B}_\epsilon} d\Gamma = 2\pi c.$$

The fundamental solution in the two-dimensional space is then found to be

$$\varphi = \frac{1}{2\pi} \ln \left(\frac{1}{|x - y|} \right).$$

□

As the name suggests, φ is fundamental to any finding any solution to the Laplace equation in the setting of boundary value problems. The fundamental solution to the Laplace equation also allows us to evaluate a harmonic function through its behavior at the boundary through Green's formula.

Corollary 2.1.6 (Green's formula). *Let D be a bounded domain and $u \in C^2(\bar{D})$ be harmonic in D , then*

$$u(x) = \int_{\partial D} \frac{\partial u}{\partial \vec{n}}(y) \varphi(x, y) - u(y) \frac{\partial \varphi}{\partial \vec{n}}(x, y) \, ds(y) \quad (2.3)$$

Now to work towards the goals of proving existence and uniqueness of the solution to BVPs with the Laplacian, we note a couple of theorems, the proofs are left to the reader and can be found in [4, 2].

Theorem 2.1.7. *Harmonic functions are analytic, i.e., each harmonic function has a local power series.*

This theorem implicitly states that any harmonic functions is infinitely differentiable at any point $x \in D$. This allows us to prove the following:

Theorem 2.1.8. *Let D be a bounded domain and $u \in C^2(D) \cap C^1(\bar{D})$ be harmonic in D s.t.*

$$u = \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \Gamma,$$

for open subset $\Gamma \subset \partial D$, then $u \equiv 0$ in D .

This then lead to the classic Mean Value Theorem [4].

Theorem 2.1.9 (Mean Value Theorem). *Let u be harmonic in an open ball $B_r(x)$ and continuous in the closure $\bar{B}_r(x)$, then*

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, d\Omega = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u \, d\Gamma.$$

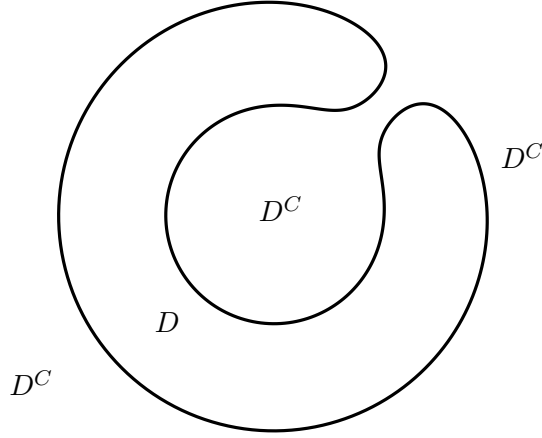


Figure 2.1: Interior, D , and exterior, D^C domain related to the C-curve

In essence, this theorem states that the function value of a harmonic function u at x is equal to the average value over $B_r(x)$ and the mean over $\partial B_r(x)$. This then leads us to the Maximum-Minimum principle [4, 2], which is vital to proving uniqueness of the solution to BVPs with the Laplacian.

Theorem 2.1.10 (Maximum-Minimum Principle). *A harmonic function u on a domain D cannot attain its maximum or minimum unless it's constant on D .*

Moreover, if u is harmonic in D and continuously extends in $\bar{D}$, then u reaches its minimum and maximum on the boundary. The theory of harmonic functions can also be extended to different types of unbounded domains. For the purpose of this survey, we are particularly interested in domains $D^c = \mathbb{R}^m \setminus \bar{D}$ with D a bounded domain itself. We refer to D as the interior domain and $\mathbb{R}^m \setminus \bar{D}$ as the exterior domain. The interior and exterior domain are shown in Figure 2.1.

We can extend Green's formula to the unbounded domain by imposing behavior at infinity [3].

Theorem 2.1.11. *Assume that D is a bounded domain of class C^1 with a connected boundary ∂D , outward unit normal $\vec{n}$ and let $u \in C^2(D^c)$ be a bounded harmonic function, then*

$$u(x) = u_\infty + \int_{\partial D} u \frac{\partial \varphi}{\partial \vec{n}} - \frac{\partial u}{\partial \vec{n}} \varphi \, d\Gamma,$$

for $x \in D^c$ and some constant u_∞ . For $d = 2$,

$$\int_{\partial D} \frac{\partial u}{\partial \vec{n}} \, d\Gamma = 0,$$

and at ∞ , for R sufficiently large enough:

$$u_\infty = \frac{1}{2\pi R} \int_{|y|=R} u(y) \, ds(y)$$

2.2 Laplace BVPs

We define four classical Laplace BVPs which are the combinations of interior/exterior with Dirichlet/Neumann boundary conditions [3, 2]. Solutions to these problems are shown in Figure 2.3.

Problem 2.2.1 (Interior Dirichlet Problem). Find a function u that is harmonic in D , continuous in $\bar{D}$, and satisfies the Dirichlet boundary condition,

$$\begin{aligned} -\Delta u &= 0, & x \in D, \\ u &= f, & x \in \partial D, \end{aligned}$$

with f a given continuous function.

Problem 2.2.2 (Interior Neumann Problem). Find a function u that is harmonic in D , continuous in $\bar{D}$, and satisfies the Neumann boundary condition,

$$\begin{aligned} -\Delta u &= 0, & x \in D, \\ \frac{\partial u}{\partial \vec{n}} &= g, & x \in \partial D, \end{aligned}$$

in the sense

$$\lim_{h \rightarrow +0} \vec{n}(x) \cdot \nabla u(x - h\vec{n}(x)) = g(x), \quad x \in \partial D,$$

uniformly, with g a given continuous function.

Problem 2.2.3 (Exterior Dirichlet Problem). Find a function u that is harmonic in D^C , continuous in $\bar{D}^C$, and satisfies the Dirichlet boundary condition,

$$\begin{aligned} -\Delta u &= 0, & x \in D^C, \\ u &= f, & x \in \partial D, \end{aligned}$$

with f a given continuous function and

$$u(x) = \begin{cases} \mathcal{O}(1), & d = 2, \\ o(1), & d \geq 3, \end{cases}$$

uniformly.

Problem 2.2.4 (Exterior Neumann Problem). Find a function u that is harmonic in D^C , continuous in $\bar{D}^C$, and satisfies the Neumann boundary condition,

$$\begin{aligned} -\Delta u &= 0, & x \in D^C, \\ \frac{\partial u}{\partial \vec{n}} &= g, & x \in \partial D, \end{aligned}$$

and

$$u(x) = o(1)$$

uniformly. The solution u converges to the boundary in the sense

$$\lim_{h \rightarrow +0} \vec{n}(x) \cdot \nabla u(x + h\vec{n}(x)) = g(x), \quad x \in \partial D,$$

uniformly, with g a given continuous function.

2.3 Solution Uniqueness

We state and loosely prove uniqueness statements of these four problems. It is emphasized that uniqueness of a solution means at most 1 solution exists.

Theorem 2.3.1. *The interior and Exterior Dirichlet problems, 2.2.1 and 2.2.3, have at most 1 solution.*

Proof. Assume multiple solutions exist and they are not unique. For the interior problem with Dirichlet boundary condition f , take 2 solutions $u_1, u_2 \in C^2(D) \cap C^1(\bar{D})$, then $u = u_2 - u_1$ is harmonic with boundary values $u = 0$ on ∂D . Since u is constant on the boundary and it cannot attain its maximum or minimum in D , it must be constant by the Maximum-Minimum principle. As u extends continuously in $\bar{D}$, we have that $u \equiv 0$, thus $u_1 = u_2$. The exterior domain follows similar structure, but the behavior at infinity plays an important role.

If a solution to the interior/exterior Dirichlet Laplace problem exists, it is unique. $\square$

More interesting things happen when the problems do not have a unique solution, but the null-space of the problem is non-trivial, as is the case for the interior Neumann problems. For the purpose of the proof, we define parallel surfaces.

Definition 2.3.2 (Parallel surfaces). Let D be a domain of class C^1 and $\vec{n}$ be the outward unit normal vector on the boundary. The parallel surface of ∂D is defined as

$$\partial D_h := \{z = x + h\vec{n}(x) \mid x \in \partial D\}$$

Theorem 2.3.3. *The interior Neumann problem, 2.2.2, have unique solutions up to a constant. The exterior Neumann problem, 2.2.4, has at most 1 solution.*

Proof. We first prove the result on the interior domain. Assume multiple solutions exist and are not unique. Let u_1 and u_2 be solutions to 2.2.2, let $u = u_1 - u_2$. The function u solve 2.2.2 with $g \equiv 0$. Suppose u is not constant on D , then there exists a set $B \subset D$ s.t.

$$\int_B |\nabla u|^2 \, d\Omega > 0.$$

Take h small enough such that $B \subset D_{-h}$, then using Green's first theorem, we see that

$$\int_B |\nabla u|^2 \, d\Omega \leq \int_{D_{-h}} |\nabla u|^2 \, d\Omega = \int_{\partial D_{-h}} u \frac{\partial u}{\partial \vec{n}} \, d\Gamma.$$

Taking $h \rightarrow 0$, since $u \in C^1(\bar{D})$, i.e., we have that

$$0 < \int_B |\nabla u|^2 \, d\Omega \leq 0,$$

A contradiction, therefore u is constant. No further restrictions are applied to u , since $\frac{\partial}{\partial \vec{n}}$ invariant to translations by a constant, u_1 and u_2 differ by a constant.

For the exterior problem, we can work through it similarly, yet $u_\infty = 0$, since $u = o(1)$. This acts as the constant value for u , hence $u_1 \equiv u_2$. If a solution to the exterior Neumann Laplace problem exists, it is unique. $\square$

2.4 Solution Existence

The existence proofs are very informative, particularly for numerical methods to solving the Laplace equations through boundary integral equations [3, 9]. For the purpose of existence proofs, we first discuss layer potentials which play a vital role in BIE methods and the rest of this overview. Afterwards, solutions to the Laplace equations are found in terms of these layer potentials by looking at their behavior near the boundary of D from both sides.

2.4.1 Layer Potentials

We first define two integral operators vital to the rest of this survey [3, 8].

Definition 2.4.1 (Single- and double-layer operators). Given a density function $\tau \in C(\partial D)$, the integral operators

$$\mathcal{S}\tau = \int_{\partial D} s(\cdot, y)\tau(y) \, ds(y),$$

and

$$\mathcal{D}\tau = \int_{\partial D} d(\cdot, y)\tau(y) \, ds(y),$$

are called, respectively, the single- and double-layer operators. The single- and double-layer kernels are respectively defined as

$$s(x, y) = \varphi(x, y) = \frac{1}{2\pi} \ln \left(\frac{1}{|x - y|} \right)$$

and

$$d(x, y) = \frac{\partial \varphi}{\partial \vec{n}(y)}(x, y) = \frac{1}{2\pi} \frac{\vec{n}(y) \cdot (x - y)}{|x - y|^2},$$

with φ the fundamental solution. The double- and single-layer kernels on the C-curve with target $x = 0$ are shown in Figure 2.2.

We can now define the single- and double layer potentials as

Definition 2.4.2 (Single- and double-layer potentials). Given a density function $\tau \in C(\partial D)$, we have

$$u(x) = [\mathcal{S}\tau](x),$$

is called the single-layer potential and

$$v(x) = [\mathcal{D}\tau](x),$$

is called the double-layer potential.

Using the Leibniz Integral Rule, we see that both the single- and double-layer potentials are harmonic functions on domain $\mathbb{R}^m \setminus \partial D$. Since they are harmonic functions, there might exist specific density function(s) τ such that u and v satisfy the Laplace BVPs with specific boundary data! Indeed we will prove for a class of boundary data, when a density function can be found, leading to boundary integral equations and the existence of solutions.

Before existence, we touch on how to interpret the single- and double-layer potentials through sources, targets and charges, common terminology in BIEs in scattering theory [9, 1]. The

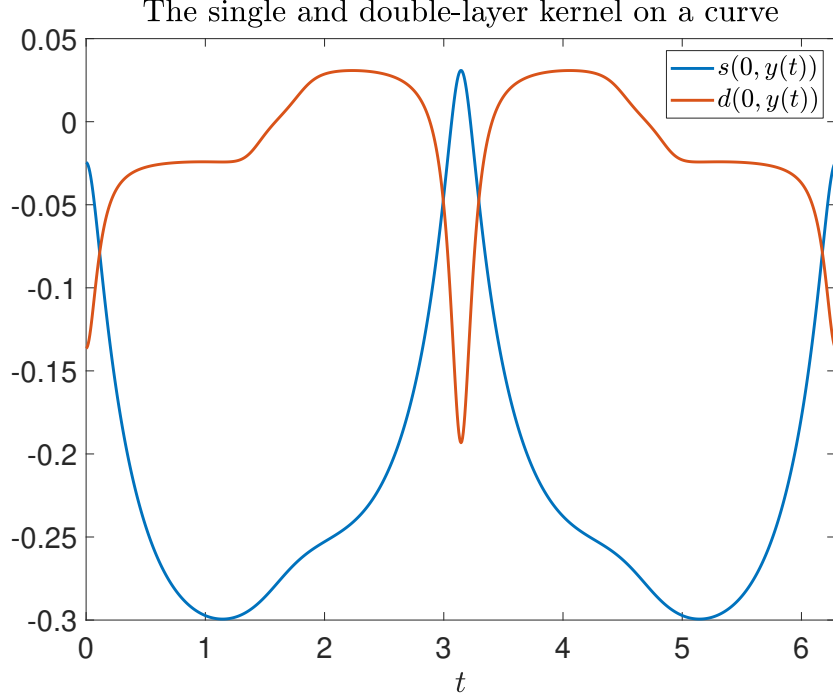


Figure 2.2: The single and double-layer kernel on the C-curve with target $x = 0$.

evaluate single-layer potential u at a *target point* x , we approximate the single layer integral operator using a quadrature scheme

$$u(x) \approx \sum_j \varphi(x, y_j) \tau(y_j) w(y_j),$$

where $w(y_j)$ includes quadrature weights and the geometry the boundary ∂D . The value of u at a target point x is a sum of various charges coming from *source points* y_j with charge $\varphi(x, y_j) \tau(y_j) w(y_j)$. Since, φ has a singularity when the target and source points are the same, we look at this as if there is a pole, the source points, that radiates a charge and affects the value of u at the target point. For this reason, we also call the source point a monopole in this setting. The charge is determined on the distance between x and y_j through φ with a boundary ∂D dependent initial charge from $w(y_j)$ and $\tau(y_j)$ describes the charge coming from the value of u on the boundary itself ∂D . This all combines to one charge coming from one monopole, the full value of u at the target point is then the sum of all source points with their respective charges. The true single-layer potential then describes the charges coming from an infinite amount of source points.

The double layer potential v at a target point x can also be approximated through a quadrature scheme,

$$v(x) \approx \sum_j \frac{\partial \varphi}{\partial \vec{n}(y_j)}(x, y_j) \tau(y_j) w(y_j).$$

To obtain a conceptual grasp, we also approximate the normal derivative through a finite difference formula, for example centered difference,

$$\frac{\partial \varphi}{\partial \vec{n}(y_j)}(x, y_j) \approx \frac{\varphi(x, y_j + h \vec{n}(y_j)) - \varphi(x, y_j - h \vec{n}(y_j))}{2h},$$

for h small. The approximation to the double-layer potential,

$$v(x) \approx \frac{1}{h} \sum_j [\varphi(x, y_j + h\vec{n}(y_j))\tau(y_j)w(y_j) - \varphi(x, y_j - h\vec{n}(y_j))\tau(y_j)w(y_j)],$$

can then be interpreted as the affect of two monopoles with charges working in opposite directions and scaled by h , hence y_j is referred to as a dipole in this setting. The double-layer potential can be seen as vibrations. [WV: Rework.](#)[Add figures on the mono- and dipole.](#)

2.4.2 Jump conditions

To use the single- and double-layer potentials to solve interior and exterior Laplace problems, a good grasp on the limiting behavior of $u(x)$ and $v(x)$ as x reaches a point on the boundary ∂D . To do so, we first discuss the behavior of the double-layer kernel as x reaches a point on ∂D over ∂D . Then we look at the behavior of the single- and double-layer potentials behavior coming from the interior and exterior domain.

Let $\gamma : [0, L] \rightarrow \partial D : t \mapsto \gamma(t)$ be an arclength parametrization of ∂D , $x(s) = \gamma(s)$, $y(t) = \gamma(t)$, and $n(t)$ the outward normal, with Frenet relation $\gamma''(t) = \kappa(t) \cdot n(t)$, where κ is the curvature of the boundary. We have L the arc length and $|\gamma'| \equiv 1$. Taylor expanding the parametrization at t ,

$$\gamma(t + \varepsilon) - \gamma(t) = \varepsilon \gamma'(t) + \frac{\varepsilon^2}{2} \gamma''(t) + O(\varepsilon^3).$$

Since $n(t) \cdot \gamma'(t) = 0$, by definition, and $n(t) \cdot \gamma''(t) = \kappa(t)$,

$$n(t) \cdot (x(s) - y(t)) = \frac{\varepsilon^2}{2} \kappa(t) + O(\varepsilon^3),$$

and

$$|x(t + \varepsilon) - y(t)|^2 = (x(t + \varepsilon) - y(t)) \cdot (x(t + \varepsilon) - y(t)) = \varepsilon^2 |y'(t)| + O(\varepsilon^3).$$

Therefore, since $|\gamma'| \equiv 1$,

$$d(t + \varepsilon, t) = \frac{1}{2\pi} \cdot \frac{\frac{\varepsilon^2}{2} \kappa(t) + O(\varepsilon^3)}{\varepsilon^2 + O(\varepsilon^3)}.$$

Taking $\varepsilon \rightarrow 0$,

$$d(t, t) = \frac{\kappa(t)}{4\pi}.$$

The kernel has a removable singularity with the limit found. The single-layer kernel has no such limit. This difference leads us to the core engine of boundary integral equations, the jump of the double layer operator from the exterior domain to the interior domain [3]. First we prove Gauss' formula

Lemma 2.4.3 (Gauss' formula). *Let D be a bounded domain of class C^2 and $D^C := \mathbb{R}^m \setminus \bar{D}$. We have that*

$$p(x) = [\mathcal{D}1](x) = \int_{\partial D} \frac{\partial \varphi}{\partial \vec{n}(y)}(x, y) \, ds(y) = \begin{cases} -1, & x \in D \\ \frac{-1}{2}, & x \in \partial D \\ 0, & x \in D^C \end{cases} \quad (2.4)$$

Proof. For $x \in D$, Green's formula shows

$$u = \int_{\partial D} \frac{\partial u}{\partial \vec{n}} \varphi - u \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma, \quad \text{for } x \in D.$$

Taking $u \equiv 1$ shows the first case.

For $x \in D^C$, Gauss' first theorem shows

$$\int_D u \Delta \varphi + \nabla u \cdot \nabla \varphi \, d\Omega = \int_{\partial D} u \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma.$$

Taking $u \equiv 1$ and the fact that $\varphi(x, y)$ is harmonic in y for $x \notin D$, shows

$$\int_{\partial D} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma = 0$$

Lastly, taking $x \in \partial D$ requires a little more work. Fix $x \in \partial D$, let $D_\epsilon := D \setminus B_\epsilon(x)$, $\partial \tilde{B}_\epsilon = \{y \mid |y - x| = \epsilon \text{ and } y \in D\}$ and $\partial \tilde{D}_\epsilon = \{y \in \partial D \mid |x - y| > \epsilon\}$. From the previous result, we know that

$$\int_{\partial \tilde{B}_\epsilon} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma + \int_{\partial \tilde{D}_\epsilon} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma = \int_{\partial D_\epsilon} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma = 0.$$

Since $\partial \tilde{D}_\epsilon \rightarrow \partial D$ as $\epsilon \rightarrow 0$, we find $\int_{\partial \tilde{B}_\epsilon} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma$ as $\epsilon \rightarrow 0$. In the two-dimensional setting, since the boundary is assumed to be smooth, we have that $\tilde{B}_\epsilon$ tends to the arc of a semi-circle. By parameterizing the arc semi-circle, we find that

$$\int_{\partial \tilde{B}_\epsilon} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma = \frac{1}{2}, \quad \forall \epsilon > 0$$

Therefore,

$$\int_{\partial \tilde{D}_\epsilon} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma = -\frac{1}{2}, \quad \forall \epsilon.$$

Taking $\epsilon \rightarrow 0$, we see that

$$\int_{\partial D} \frac{\partial \varphi}{\partial \vec{n}} \, d\Gamma = -\frac{1}{2}$$

□

WE now have all the tools to find the jump of the double layer potential [3, 8].

Theorem 2.4.4 (The double-layer potential jump). *For ∂D of class C^2 , the double-layer potential v with continuous density function τ can be continuously extended from D to $\bar{D}$ and from D^C to $\overline{D^C}$ with*

$$v_\pm(x) = \pm \frac{1}{2} \tau(x) + [\mathcal{D}\tau](x), \quad (2.5)$$

for $x \in \partial D$, where $v_\pm(x) := \lim_{h \rightarrow +0} v(x \pm h\vec{n}(x))$.

Proof. The proof is through the classical splitting technique. Fix $x_0 \in \partial D$, we have

$$\begin{aligned} [\mathcal{D}\tau](x) &= \int_{\partial D} d(x, y) \tau(y) \, ds(y), \\ &= \underbrace{\int_{\partial D} d(x, y) (\tau(y) - \tau(x_0)) \, ds(y)}_{\tilde{v}(x)} + \tau(x_0) p(x). \end{aligned} \quad (2.6)$$

The rest of the proof depends on proving that $\tilde{v}(x)$ is continuous in the normal direction over x_0 and finding the value at x_0 . To this end, let $x_h = x_0 + h\vec{n}(x_0)$, $\Gamma_\delta = \partial D \cap B_\delta(x_0)$ and $\Gamma_\delta^C = \partial D \setminus \Gamma_\delta$. For $y \in \Gamma_\delta^C$, $d(x, y)$ is continuous in $h \rightarrow 0$. If we thus show that the integral over Γ_δ vanishes uniformly in h , we have shown that $\tilde{v}$ is continuous.

Without loss of generality, let $x_0 = 0$ and $\vec{n}(x_0) = (0, 1)^T$ (shift and rotation invariances), approximate the tangent line with $(s, g(s))$ where $g(0) = 0$ and $g'(0) = 0$. Since ∂D is of the class C^2 , we find through Taylor approximations that $g(s) = \frac{1}{2}\kappa(0)s^2 + \mathcal{O}(s^3)$, where $\kappa(0)$ is the curvature. Therefore we can bound g for s near 0,

$$|g(s)| \leq C|s^2| \text{ and } |g'(s)| \leq C|s|.$$

Now take $x = (0, h)$, on the tangent and y goes over $(s, g(s))$ as Γ_δ takes $\delta \rightarrow 0$. We then have

$$x - y = (-s, h - g(s)), \Rightarrow |x - y|^2 = s^2 + h^2 - 2hg(s) + g(s)^2 = s^2 + h^2 + \mathcal{O}(s^2h + s^4).$$

Therefore,

$$\vec{n}(y) = \frac{(-g'(s), 1)}{\sqrt{1 + g'(s)^2}} \Rightarrow \vec{n}(y) \cdot (x - y) = \frac{sg'(s) - g(s) + h}{\sqrt{1 + g'(s)^2}}.$$

Since $sg'(s) - g(s) = \mathcal{O}(s^2)$ and $1 + g'(s)^2 = 1 + \mathcal{O}(s^2)$, we have that

$$\vec{n}(y) \cdot (x - y) = h + \mathcal{O}(s^2).$$

Under the assumption that the density function τ is Hölder continuous, which is typically granted for (piecewise) smooth kernels,

$$\underbrace{|\tau(y) - \tau(x_0)|}_{\text{Hölder Continuity}} \leq L|y - x_0|^\alpha = L\left(\sqrt{s^2 + g^2(s)}\right)^\alpha \leq \tilde{L}|s|^\alpha,$$

we have that

$$\frac{|\vec{n}(y) \cdot (x - y)|}{|x - y|^2} |\tau(y) - \tau(x_0)| \leq \frac{|h| + Cs^2}{s^2 + h^2} |s|^\alpha.$$

Plugging this into the integral over Γ_δ , gives us

$$\int_{\Gamma_\delta} \frac{|\vec{n}(y) \cdot (x - y)|}{|x - y|^2} |\tau(y) - \tau(x_0)| ds(y) \leq \int_{|s| \leq \delta} \frac{h + Cs^2}{s^2 + h^2} |s|^\alpha ds,$$

where

$$\int_{|s| < \delta} \frac{h}{s^2 + h^2} |s|^\alpha ds \leq \delta^\alpha \int_{-\infty}^{\infty} \frac{h}{s^2 + h^2} ds = \pi\delta^\alpha,$$

and

$$\int_{|s| < \delta} \frac{Cs^2}{s^2 + h^2} |s|^\alpha ds \leq C \int_{-\delta}^{\delta} |s|^\alpha ds = \frac{2C}{2 + \alpha} \delta^{1+\alpha}.$$

Therefore,

$$\int_{\Gamma_\delta} \frac{|\vec{n}(y) \cdot (x - y)|}{|x - y|^2} |\tau(y) - \tau(x_0)| ds(y) \leq \tilde{C}\delta^\alpha,$$

uniformly in h , thus $\tilde{v}(x_h) \xrightarrow{h \rightarrow 0} \tilde{v}(x_0)$. We now find this $v(x_0)$ and show the prove the statement. Use the Gauss' formula (2.4), we know that $p(x_0) = \frac{-1}{2}$, thus from (2.6) we know that

$$[\mathcal{D}\tau](x_0) = v(x_0) = \tilde{v}(x_0) - \frac{1}{2}\tau(x_0) \Rightarrow \tilde{v}(x_0) = v(x_0) + \frac{1}{2}\tau(x_0) = \frac{1}{2}\tau(x_0) + [\mathcal{D}\tau](x_0).$$

We also know from the Gauss' formula the limiting behavior, i.e.,

$$v_+(x_0) = \lim_{h \rightarrow +0} v(x_h) = \tilde{v}(x_0) + 0\tau(x_0) = \tilde{v}(x_0),$$

and

$$v_-(x_0) = \lim_{h \rightarrow -0} v(x_h) = \tilde{v}(x_0) - \tau(x_0).$$

This leads us to an equation that will be used later,

$$v_+(x) - v_-(x) = \tau(x) \quad \text{for } x \in \partial D, \quad (2.7)$$

and the statement in the theorem,

$$v_+(x_0) = \frac{1}{2}\tau(x_0) + [\mathcal{D}\tau](x_0),$$

and

$$v_-(x_0) = \frac{1}{2}\tau(x_0) + [\mathcal{D}\tau](x_0) - \tau(x_0) = -\frac{1}{2}\tau(x_0) + [\mathcal{D}\tau](x_0).$$

□

For the single layer potential, no jump occurs due to the weak singularity. However, it can be used for the interior/exterior Neumann problems. Let $u(x) = [\mathcal{S}\tau](x)$, we then have that

$$\frac{\partial}{\partial \vec{n}(x)} u(x) = \frac{\partial}{\partial \vec{n}(x)} [\mathcal{S}\tau](x) = \int_{\partial D} \frac{\partial}{\partial \vec{n}(x)} \varphi(x, y) \tau(y) \, ds(y).$$

We define the integral operator to be the adjoint of the double layer potential,

$$\mathcal{D}'\tau = \int_{\partial D} \frac{\partial}{\partial \vec{n}(\cdot)} \varphi(\cdot, y) \tau(y) \, ds. \quad (2.8)$$

Note that this is an adjoint w.r.t. the bilinear inner product $\langle \cdot, \cdot \rangle$, and it itself is not a layer potential since it cannot satisfy any Laplace equation. However, similar jump conditions are observed, making them useful for the Neumann problems. Given $u(x) = [\mathcal{S}\tau](x)$, we can prove

$$\frac{\partial u_+}{\partial \vec{n}(x_0)}(x_0) = -\frac{1}{2}\tau(x_0) + [\mathcal{D}'\tau](x_0), \quad (2.9)$$

and

$$\frac{\partial u_-}{\partial \vec{n}(x_0)}(x_0) = +\frac{1}{2}\tau(x_0) + [\mathcal{D}'\tau](x_0). \quad (2.10)$$

This jump is special to the usage of the single layer potential. The jump condition of the double-layer operator does not survive the normal derivative and thus no jump condition is seen for the double-layer potential. In particular,

$$\frac{\partial v_+}{\partial \vec{n}} - \frac{\partial v_-}{\partial \vec{n}} = 0 \quad \text{on } \partial, \quad (2.11)$$

On the other hand for the single layer potential, since it observes no jump we have that

$$u_+ - u_- = 0 \quad \text{on } \partial, \quad (2.12)$$

However, the normal of the single layer potential does observe this jump. Using (2.9) and (2.10), we get

$$\frac{\partial u_+}{\partial \vec{n}} - \frac{\partial u_-}{\partial \vec{n}} = -\tau \quad \text{on } \partial D, \quad (2.13)$$

In the interest of future endeavors, we mention two facts on these kernels and future similar kernels [8].

First, we note that if ∂D is of the class C^2 , there exists constants L_1 and L_2 , s.t.,

$$|\vec{n}(x) \cdot (x - y)| \leq L_1 |x - y|^2 \text{ and } |\vec{n}(x) - \vec{n}(y)| \leq L_2 |x - y|,$$

which is seen through Taylor approximations where L_1 and L_2 are similar to the curvature. This can then be used to see that the double-layer kernel is weakly singular in dimension $d \geq 2$,

$$|d(x, y)| \sim \frac{|\vec{n}(x) \cdot (x - y)|}{|x - y|^d} \leq \frac{\tilde{L}}{|x - y|^{d-2}} \text{ for } x, y \in \partial D,$$

hence $\mathcal{D}$, and $\mathcal{D}'$ are compact operators! It is trivial to see that the single layer kernel is also weakly singular and hence $\mathcal{S}$ is also a compact operator.

Second, with scattering problems in sight, in particular with the Helmholtz equation, we have to note that the fundamental solution to the Helmholtz equation has similar singular behavior at $|x - y| \rightarrow 0$, therefore these jump conditions will translate to the Helmholtz equation with new fundamental solutions.

2.4.3 Solutions through BIEs

The limiting behavior to the boundary now allows to us to reformulate the PDE into a BIE. For the Dirichlet Laplace problems we will be using the jump conditions described in Theorem 2.4.4. For the Interior Dirichlet problems (2.2.1) with boundary condition $u = f$ on ∂D , we have

$$\begin{aligned} u(x) &= [\mathcal{D}\tau](x), & x \in D, \\ \left[\left(-\frac{I}{2} + \mathcal{D} \right) \tau \right] (x) &= f(x), & x \in \partial D. \end{aligned} \quad (2.14)$$

For the Exterior Dirichlet problem (2.2.3) with boundary condition $u = f$ on ∂D , we have

$$\begin{aligned} u(x) &= [\mathcal{D}\tau](x), & x \in D, \\ \left[\left(\frac{I}{2} + \mathcal{D} \right) \tau \right] (x) &= f(x), & x \in \partial D. \end{aligned} \quad (2.15)$$

We can also reformulate the Neumann problems to BIEs using the jump conditions (2.10) and (2.9). For the Interior Neumann problem (2.2.2) with boundary condition $\frac{\partial u}{\partial \vec{n}} = g$ on ∂D , we have

$$\begin{aligned} u(x) &= [\mathcal{S}\tau](x), & x \in D, \\ \left[\left(\frac{I}{2} + \mathcal{D}' \right) \tau \right] (x) &= g(x), & x \in \partial D. \end{aligned} \quad (2.16)$$

For the Exterior Neumann problem (2.2.4) with boundary condition $\frac{\partial u}{\partial \vec{n}} = g$ on ∂D , we have

$$\begin{aligned} u(x) &= [\mathcal{S}\tau](x), & x \in D, \\ \left[\left(-\frac{I}{2} + \mathcal{D}' \right) \tau \right] (x) &= g(x), & x \in \partial D. \end{aligned} \quad (2.17)$$

If we are able to find the density layer function, we are able to find solutions to the Helmholtz equation and thus prove the existence of solutions to the Helmholtz equation. To this end, we have to be able to invert operators like $\pm \frac{I}{2} + \mathcal{D}$ and $\pm \frac{I}{2} + \mathcal{D}'$. There are "direct" boundary integral formulations that, given Dirichlet boundary data, find Neumann boundary data to then use the Green's formula (2.3). For example, for the Dirichlet problem, we find the Neumann boundary data through

$$\int_{\partial D} s(x, y) \frac{\partial}{\partial \vec{n}(y)} u(y) \, ds(y) = \frac{1}{2} f(x) + \int_{\partial D} d(x, y) f(y) \, ds(y), \quad x \in \partial D, \quad (2.18)$$

or given Neumann boundary data, we find the Dirichlet boundary data through

$$\frac{1}{2} u(x) + \int_{\partial D} d(x, y) u(y) \, ds(y) = \int_{\partial D} s(x, y) g(y) \, ds(y). \quad (2.19)$$

While the two do not need to an extra density functions τ , which comes with it's extra problems, we instead compute the other boundary data. However, an operator is still needs to be inverted and evaluated. In particular, here we must invert a compact operator, or Fredholm integral equation of the first kind $\mathcal{S} \frac{\partial u}{\partial \vec{n}}$. On the other hand, since $\mathcal{D}$ and $\mathcal{D}'$ are compact operators, we have that $\pm \frac{I}{2} + \mathcal{D}$ and $\pm \frac{I}{2} + \mathcal{D}'$ Fredholm integral equations of the second kind on which we are able to apply the Fredholm alternative!

Theorem 2.4.5. *Let $\mathcal{D}$ and $\mathcal{D}'$ be the double-layer operator and adjoint to the double-layer operator, respectively, then*

$$\mathcal{N} \left(-\frac{I}{2} + \mathcal{D} \right) = \mathcal{N} \left(-\frac{I}{2} + \mathcal{D}' \right) = \{0\},$$

and

$$\mathcal{N} \left(\frac{I}{2} + \mathcal{D} \right) = \text{span}\{1\} \text{ and } \mathcal{N} \left(\frac{I}{2} + \mathcal{D}' \right) = \text{span}\{\psi_0\},$$

with

$$\int_{\partial D} \psi_0 \, d\Gamma = 1$$

Proof. Let τ exist s.t. $\left(\frac{I}{2} + \mathcal{D} \right) \tau \equiv 0$. We then know that the double-layer potential with this density function, $u = \mathcal{D}\tau$, is harmonic. Using the behavior at the boundary found in Theorem 2.4.4, we know that

$$u_- \equiv \left(-\frac{I}{2} + \mathcal{D} \right) \tau \equiv 0.$$

By the uniqueness of the interior Dirichlet problem, we then have that $u(x) = 0$ for $x \in D$, hence $\frac{\partial u_-}{\partial \vec{n}} \equiv 0$. Since (2.11), we have that $\frac{\partial u_+}{\partial \vec{n}} \equiv 0$ which now corresponds to the exterior Neumann problem. Due to uniqueness of the exterior Neumann solution, we know that $u \equiv 0$ for $x \in D^C$, therefore, $u_+ \equiv 0$. Therefore, because of the jump condition of the double-layer potential (2.7), we have that

$$\tau = u_+ - u_- = 0,$$

hence the operator $-\frac{I}{2} + \mathcal{D}$ is injectivity and due to the Fredholm alternative, it is invertible. Moreover, because of the Fredholm alternative,

$$\mathcal{N}\left(-\frac{I}{2} + \mathcal{D}\right) = \mathcal{N}\left(-\frac{I}{2} + \mathcal{D}'\right) = \{0\},$$

On the other hand, for the operator $\frac{I}{2} + \mathcal{D}$, we solve the exterior Dirichlet problem, for which we again can use the uniqueness property. However, to use (2.7) again, we need the solution to the interior Neumann problem to be unique. We in fact know it is not and solutions differ by a constant. Hence

$$\mathcal{N}\left(\frac{I}{2} + \mathcal{D}\right) = \text{span}\{1\}.$$

The Fredholm alternative then also tells us

$$\mathcal{N}\left(\frac{I}{2} + \mathcal{D}'\right) = \text{span}\{\psi_0\},$$

with

$$\int_{\partial D} \psi_0 \, d\Gamma = 1,$$

due to $\mathcal{D}'$ being the adjoint of $\mathcal{D}$ w.r.t. the bilinear inner-product. $\square$

The invertibility of the first two integral operators now proves existence of the solution to the interior Dirichlet problem and exterior Neumann problem. This alongside the uniqueness proofs, proves that interior Dirichlet problem and exterior Neumann problem have unique solutions. While this seems like a convoluted way to prove existence of solutions, it also gives us a very nice reformulation that can be exploited for numerical computations, as we will see in later chapters. In particular, we will numerically invert a Fredholm integral operator of the second kind, which comes with well-conditioned matrices [7, 9].

While the other operators are not injective, it does not mean there exists no solution to the associated problems! Instead, we use a different BIE, in particular a combination of the double- and single-layer operators.

Theorem 2.4.6 (“Modified” double layer potential). *Let $D \subset \mathbb{R}^m$, be a bounded domain with $y_0 \in D$ and $D^C = \mathbb{R}^m \setminus \bar{D}$,*

$$u(x) = \underbrace{\int_{\partial D} (d(x, y) + s(x, y_0)) \tau(y) \, ds(y)}_{[\mathcal{K}\tau](x)}, \quad x \in D^C \quad (2.20)$$

with $\tau \in C$ s.t.

$$\frac{\tau(x)}{2} + [\mathcal{K}\tau](x) = f(x), \quad x \in \partial D,$$

is a solution to the exterior Dirichlet problem.

Proof. Without loss of generality, assume $y_0 = 0$ with a translated domain. To prove this, we prove the invertibility of the operator $I/2 + \mathcal{K}$. First note that the kernel $d(x, y) + s(x, y)$ is weakly singular when $x = y$, therefore $\mathcal{K}$ is a compact operator and thus $(I/2 + \mathcal{K})$ is a Fredholm integral operator of the second kind. Since $\mathcal{D}$ is compact, $\mathcal{K} - \mathcal{D}$ is compact and

therefore continuous. Let $\tau \in C$ s.t. $(I/2 + \mathcal{K})\tau = 0$, $u = \mathcal{K}\tau$ solve an exterior Dirichlet problem and

$$u_+ = \left(\frac{I}{2} + \mathcal{K}\right)\tau = 0 \text{ on } \partial D,$$

therefor $u \equiv 0$ in D^C . Since $d(x, y)/s(x, 0) \rightarrow 0$ as $|x| \rightarrow \infty$, we have

$$(s(x, 0))^{-1}u \rightarrow \int_{\partial D} \tau(y) \, ds(y), \text{ as } |x| \rightarrow \infty,$$

therefore

$$\int_{\partial D} \tau(y) \, ds(y) = 0.$$

Moreover, from the boundary we know

$$(s(x, 0))^{-1}u_+(x) = \frac{1}{2}(s(x, 0))^{-1}\tau(x) + (s(x, 0))^{-1}[\mathcal{D}\tau](x) + (s(x, 0))^{-1}[(\mathcal{K} - \mathcal{D})\tau](x),$$

and we see that

$$(s(x, 0))^{-1}[(\mathcal{K} - \mathcal{D})\tau](x) = \int_{\partial D} \tau(y) \, ds(y) = 0,$$

therefore,

$$(s(x, 0))^{-1} \left(\frac{1}{2} + \mathcal{D}\right)\tau = 0 \Rightarrow \left(\frac{1}{2} + \mathcal{D}\right)\tau = 0.$$

From Theorem 2.4.5, we now know that $\tau \in \text{span}\{1\}$, yet since $\int_{\partial D} \tau d\Gamma = 0$, we have that $\tau \equiv 0$ and thus $I/2 + \mathcal{K}$ is injective. By the Fredholm alternative, $I/2 + \mathcal{K}$ is invertible and therefore (2.20) exists and is a solution to the exterior Dirichlet problem. $\square$

Lastly, we mention that the interior Neumann problem is solvable iff $\int_{\partial D} g \, d\Gamma = 0$ [3]. The necessary condition is a results from applying Green's first theorem. The exterior Neumann problem always has a unique solution in $\mathbb{R}^3$ and has a unique solution in $\mathbb{R}^2$ iff $\int_{\partial D} g \, d\Gamma = 0$ due to the behavior of the fundamental solution at infinity. Solutions using the boundary integral equations is shown in Figure 2.3.

Lastly we mention the following theorem showing continuous dependence on the boundary.

Theorem 2.4.7. *Solutions to the Dirichlet and Neumann problem depends continuously in $\|\cdot\|_\infty$ on the given boundary data.*

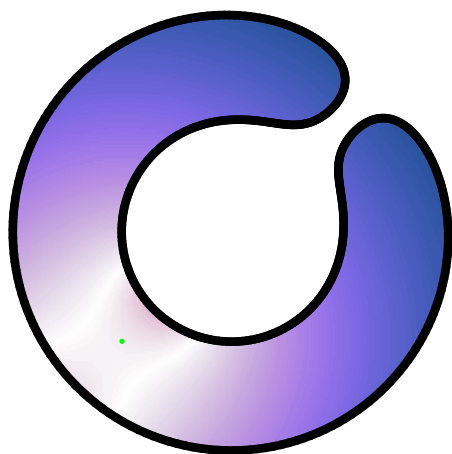
2.4.4 Notes on non-smooth domains

Throughout the section it is always assumed that the boundary is smooth enough, i.e., ∂D is of class C^2 . This can be extended to Lyapunov boundaries without much change in the previous discussions, i.e.,

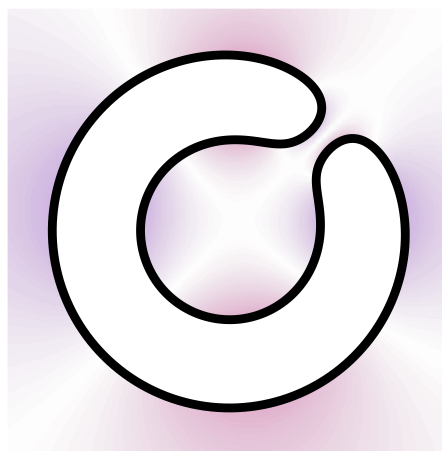
$$\forall x \in \partial D, \vec{n}(x) \text{ exists and } \vartheta(x, y) = \angle(\vec{n}(x), \vec{n}(y)) \text{ has } L, \alpha \text{ s.t. } \vartheta(x, y) \leq L|x - y|^\alpha.$$

However, many geometries of interest have corners (rectangles, triangles, keyholes, microwaves, ...) or cusps (teardrops, cardioids or manifolds with pitched ends, ...), as seen in Figure [Figure including the \$\gamma_i\$ notation](#) [1, 8]. We will not work the BIE linked to these geometries out, but these corners and cusps have major consequences. For one, the section of the boundary inside

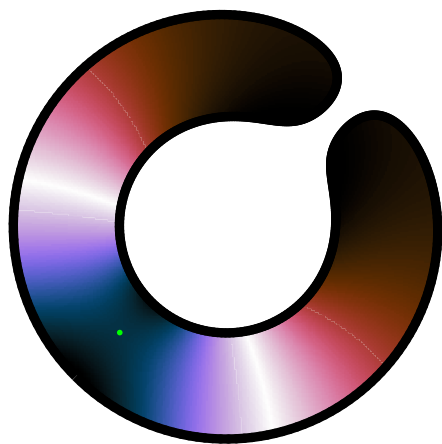
Interior Dirichlet Problem



Exterior Dirichlet Problem



Interior Neumann Problem



Exterior Neumann Problem

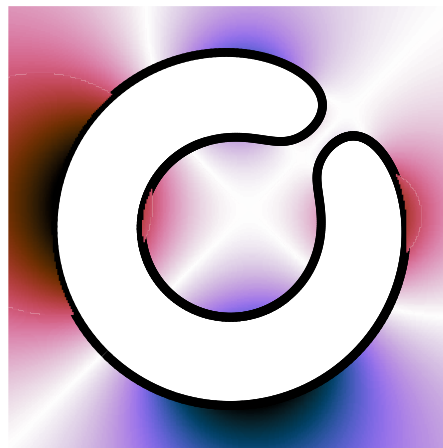


Figure 2.3: Solutions to Laplace boundary value problems. The exterior problems have boundary condition $\sin(5s)$ for $s \in [0, 2\pi)$ and the interior problems have a source point (green dot) inside the domain.

a shrinking ball centered at the corner will no-longer go to a semi-circle as $\epsilon \rightarrow 0$. This is relevant for the proof of Theorem 2.4.3. Moreover, the curve's parametrization is no-longer twice differentiable at all points, which is necessary in the proof of Theorem 2.4.4 and is an underlying issue in all jump conditions used. The proof then requires a different tactic that we will not discuss or have read. Lastly, when the geometries observe corners or cusps we can no-longer expect the density layer function smooth to any degree. As a consequence of the first problem, we can see that the jumps change at the corner points. Let x_j be the corners of the boundary, then

$$u_{\pm}(x) = \begin{cases} \pm \frac{\gamma_j^{\pm}}{2\pi} \tau(x_j) + [\mathcal{D}\tau](x_j) & x = x_j, \\ \pm \frac{1}{2} \tau(x) + [\mathcal{D}\tau](x) & x \neq x_j, \end{cases}$$

with $\gamma_j^+ = \gamma_j$ and $\gamma_j^- = 2\pi - \gamma_j$, and γ_j the interior angle at corner x_j in radians. The double layer potential can be reformulate to adapt to the corners

$$[\hat{\mathcal{D}}\tau](x) = \begin{cases} [\mathcal{D}\tau](x) + \left(\frac{\gamma_j}{\pi} - 1\right) \tau(x), & x = x_j, \\ [\mathcal{D}\tau](x), & x \neq x_j. \end{cases}$$

Now here, for $\tau \in C(\partial D)$, we have that $\mathcal{D}\tau$ is not continuous at corners, but $\hat{\mathcal{D}}\tau$ is. The existence of a solution is still true, however it is more intense to prove [8]. We now have that if the boundary data is L^2 , the density function $\tau \in L^2(\partial D)$. The limiting convergence at the boundary is now true in a weak sense, i.e.,

$$\lim_{h \rightarrow +0} \int_{\partial D} (u(y + h\vec{n}(y)) - f(y))^2 ds(y) = 0.$$

2.5 The Laplacian's spectrum

The spectrum of the Laplacian is closely related to the Helmholtz equation and plays a vital role in both its uniqueness and existence, through BIE. We first introduce the problem statements and review all characteristics of the Laplacian's spectrum needed for the following chapter. We are particularly interested in the spectrum of the Laplacian with three boundary conditions: the Dirichlet, Neumann and Impedance (or Robin) boundary conditions.

Problem 2.5.1. Let Ω be a domain, find the eigenvalues $\lambda \in \mathbb{C}$ and non-zero eigenfunctions $u : D \rightarrow \mathbb{C}$ such that:

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) & x \in D, \\ u(x) &= 0 & x \in \partial\Omega. \end{aligned}$$

Problem 2.5.2. Let Ω be a domain, find the eigenvalues $\lambda \in \mathbb{C}$ and non-zero eigenfunctions $u : D \rightarrow \mathbb{C}$ such that:

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) & x \in \Omega, \\ \frac{\partial u}{\partial \vec{n}}(x) &= 0 & x \in \partial\Omega. \end{aligned}$$

Problem 2.5.3. Let Ω be a domain, find the eigenvalues $\lambda \in \mathbb{C}$ and non-zero eigenfunctions $u : D \rightarrow \mathbb{C}$ such that:

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) & x \in \Omega, \\ \frac{\partial u}{\partial \bar{n}}(x) + \alpha u(x) &= 0 & x \in \partial\Omega, \end{aligned}$$

with non-zero impedance parameter α representing local resistance.

While the eigenvalues and corresponding eigenfunction are very useful for solving Laplace BVPs, they play a vital role in the well-posedness of BIE reformulations of the Helmholtz equation. To this end, we inspect a limited amount of information on the locations and geometry of the eigenvalues for both interior and exterior domains.

The location of the spectrum of the Laplace equation is found through the following spectral theorem [6].

Theorem 2.5.4 (Spectral theorem for self-adjoint operators). *Let $\mathcal{A}$ be a self-adjoint operator on a Hilbert space H , i.e., $\mathcal{A} = \mathcal{A}^*$, then $\sigma(\mathcal{A}) \subset \mathbb{R}$.*

The Laplacian is indeed self-adjoint with all boundary conditions (with $\alpha \in \mathbb{R}$), are self-adjoint in $L^2(\Omega)$. Symmetry of the operator is seen through Green's first theorem. Let $f, g \in C^2(\Omega)$ satisfying the homogeneous Dirichlet or Neumann boundary condition,

$$\langle \Delta f, g \rangle = \int_{\Omega} \Delta f \bar{g} \, d\Omega = - \int_{\Omega} \nabla \bar{g} \cdot \nabla f \, d\Omega = \int_{\Omega} f \overline{\Delta g} \, d\Omega = \langle f, \Delta g \rangle.$$

For the impedance boundary condition, let $f, g \in C^2(\Omega)$ satisfy the impedance boundary condition with $\alpha \in \mathbb{R}$, i.e, $\frac{\partial f}{\partial \bar{n}} = -\alpha f$ and $\bar{g} = -\frac{1}{\alpha} \frac{\partial \bar{g}}{\partial \bar{n}}$

$$\langle \Delta f, g \rangle = \int_{\Omega} \Delta f \bar{g} \, d\Omega = - \int_{\Omega} \nabla \bar{g} \cdot \nabla f \, d\Omega + \int_{\partial\Omega} \bar{g} \frac{\partial f}{\partial \bar{n}} \, d\Gamma = - \int_{\Omega} \nabla \bar{g} \cdot \nabla f \, d\Omega + \int_{\partial\Omega} \frac{\partial \bar{g}}{\partial \bar{n}} f \, d\Gamma = \int_{\Omega} f \overline{\Delta g} \, d\Omega = \langle f, \Delta g \rangle.$$

Proving $\Delta \pm i$ is surjective to complete the self-adjoint-ness proof is outside the scope of class [6]. The spectrum is therefore real and we can even say something about its sign. For the Dirichlet and Neumann problem, if we multiply $-\Delta u = \lambda u$ by $\bar{u}$ and integrate over Ω , using Green's first theorem, we see that

$$\lambda \int_{\Omega} |u|^2 \, d\Omega = \int_{\Omega} |\nabla u|^2 \, d\Omega.$$

Since both integral are positive, to have a non-trivial solution, we must have that $\lambda > 0$. Similarly for the impedance problem, we have

$$\lambda \int_{\Omega} |u|^2 \, d\Omega = \int_{\Omega} |\nabla u|^2 \, d\Omega - \int_{\partial\Omega} \bar{u} \frac{\partial u}{\partial \bar{n}} \, d\Gamma = \int_{\Omega} |\nabla u|^2 \, d\Omega + \alpha \int_{\partial\Omega} |u|^2 \, d\Gamma.$$

If α is real and positive, the integrals and the right hand side are positive. In order to have a non-trivial eigenfunction we again must have that $\lambda > 0$.

We are also able to state if the spectrum is discrete, continuous or a mix, through theory on compact resolvents.

Definition 2.5.5 (Compact Resolvent). A closed, densely defined operator $\mathcal{A}$ on a Hilbert space H has compact resolvent if for some (equivalently, by the resolvent identity, every) $\lambda \in \rho(\mathcal{A})$, the resolvent operator $(\mathcal{A} - \lambda I)^{-1} : H \rightarrow H$ is compact.

In particular, the Riesz–Schauder theorem [6, 8] states that if an operators resolvent is compact, it has a discrete spectrum.

Theorem 2.5.6 (Riesz–Schauder theorem). *If $\mathcal{K}$ is a compact operator on a Hilbert space H , then $\sigma(\mathcal{K}) \setminus \{0\}$ consists of isolated eigenvalues of finite algebraic multiplicity, with no other spectrum (continuous or residual) away from 0, and 0 is the only possible accumulations point.*

The proof of the theorem and the fact that Δ , the Laplacian with considered boundary conditions, is closed is outside the scope of this report. If we use Riesz–Schauder theorem on $\mathcal{K} = (\Delta - \lambda_0)^{-1}$, with $\lambda_0 \in \rho(\Delta)$, it can be shown that $\sigma(\Delta)$ is discrete with isolated eigenvalues of finite algebraic multiplicity, accumulating only at ∞ . The Laplacian with given boundary conditions has a compact resolvent when the domain Ω is has a compact resolvent. This is conceptually reasoned: on a bounded domain Ω , compactness of the resolvent is essentially automatic for any elliptic operator with a reasonable domain geometry. The resolvent behaves like a bounded solution operator into $H^1(\Omega)$ composed with the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$, and this inclusion is compact precisely because Ω is bounded. This has been shown through the Rellich-Kondrachov theorem and fails on unbounded domains [4, 6]. Intuitively, a bounded H^1 sequence has its gradient bounded and lives in a fixed finite region, so it cannot escape or oscillate its way out of precompactness in L^2 . However, on an unbounded domain, this compactness fails. A bounded sequence in $H^1(\mathbb{R}^n)$ can be a sequence of translations of a fixed bump functions shifted off to infinity, which is bounded in H^1 but admits no convergent subsequence in $L^2(\mathbb{R}^n)$. As a result, on unbounded domains the Laplacian (with any boundary condition) typically has some continuous spectrum. Here, $\sigma(-\Delta) = [0, \infty)$ on $\Omega \subset \mathbb{R}^d$. This leads to the following two observations. **We should add figures of all of the spectra.**

Observation 2.5.7. The Laplacian with Dirichlet, Neumann or Impedance, with real impedance parameter, boundary conditions on the interior domain have real discrete spectrum only accumulating at infinity.

Observation 2.5.8. The Laplacian with Dirichlet, Neumann or Impedance, with real impedance parameter, boundary conditions on the exterior domain have real continuous spectrum.

Lastly, we further discuss the impedance problem with a purely imaginary impedance parameter.

Problem 2.5.9. Let Ω be a domain, find the eigenvalues $\lambda \in \mathbb{C}$ and non-zero eigenfunctions $u : \Omega \rightarrow \mathbb{C}$ such that:

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) & x \in \Omega, \\ \partial_n u(x) - i\eta u(x) &= 0 & x \in \partial\Omega, \end{aligned}$$

with real non-zero impedance parameter η representing local resistance.

It can be shown that this problems again has a compact resolvent for the associated operator, exactly as in the real- α case [4], so by the Riesz–Schauder theorem [6] the spectrum $\sigma(A)$ remains discrete. However, the symmetry of the operator does not hold, it is not self-adjoint and thus the eigenvalues are no-longer real valued. The location of the eigenvalues can still be found and it depends on the sign of η . Multiply $-\Delta u = \lambda u$ by $\bar{u}$, integrate over the domain Ω and apply Green’s identity with the impedance boundary condition,

$$\lambda \int_{\Omega} |u|^2 \, d\Omega = \int_{\Omega} |\nabla u|^2 \, d\Omega - \int_{\partial\Omega} \bar{u} \frac{\partial u}{\partial \bar{n}} d\Gamma = \int_{\Omega} |\nabla u|^2 \, d\Omega - i\eta \int_{\partial\Omega} |u|^2 d\Gamma.$$

Let $\lambda = \lambda_r + i\lambda_i$ with $\lambda_r, \lambda_i \in \mathbb{R}$. Since all integral are real valued, taking the real part of both side tells us

$$\lambda_r \int_{\Omega} |u|^2 \, d\Omega = \int_{\Omega} |\nabla u|^2 \, d\Omega, \quad (2.21)$$

since both integral are positive, in order to obtain a non-trivial eigenfunction, we must have that $\lambda_r \geq 0$. Moreover, strict inequality holds $\lambda_r > 0$, if $\lambda_r = 0$, $\nabla u = 0$, i.e., u would have to be constant. In fact, the boundary condition would be $u = 0$, hence $u \equiv 0$ in the domain and thus u would not be an eigenfunction. Taking the imaginary part we see that

$$\lambda_i \int_{\Omega} |u|^2 \, d\Omega = -\eta \int_{\partial\Omega} |u|^2 \, d\Gamma. \quad (2.22)$$

Since both integrals are positive, in order to have a non-trivial eigenfunction, we must have that λ_i and η have the opposite sign, i.e., if the η is negative, the eigenvalues will be in the upper-half-plane and if η is positive, the eigenvalues will be in the lower-half-plane. Note that λ_i cannot be real as then the eigenfunction would again be forced to be trivial. We thus have the concluding observation. **Add figures of these spectra for different eta (sign and size)**

Observation 2.5.10. The Laplacian with Impedance boundary condition with impedance parameter $i\eta$ has a discrete spectrum in the upper-half-plane when $\eta < 0$ and lower-half-plane when $\eta > 0$.

Chapter 3

Helmholtz Equation through BIE

The Helmholtz equation is what the Laplace equation becomes once we let things vibrate. It is the time-harmonic reduction of the wave equation, and it governs acoustic scattering, which is our real target. Much of the machinery carries over unchanged: the fundamental solution is singular in exactly the same way, so the layer potentials and their jump conditions are the ones we already know. What is new is how the problem lives on an unbounded exterior domain. We need the Sommerfeld radiation condition to pin down which solution we mean, and the wave number k now sits inside the operators we are trying to invert. We first state the sound-soft and sound-hard scattering problems and prove uniqueness of their solutions. We then reformulate them as boundary integral equations and apply the Fredholm alternative, which gives existence, but only for wave numbers that are not eigenvalues of an interior problem. This is a genuine failure and not an artifact of the analysis being loose: at those wave numbers the boundary integral operator really is singular, even though the scattering problem itself is perfectly well-posed. Since we cannot know where these eigenvalues sit for a general scatterer, and numerically we must avoid a whole neighborhood of each, this makes the formulation unpredictable in practice. The remedy is to combine the single- and double-layer potentials into one equation, the combined field integral equation, whose failure modes are governed by the impedance eigenvalue problem from the previous chapter and therefore sit off the real line entirely. We close by choosing the coupling parameter to push those spurious modes as far from the real axis as conditioning allows, and by assembling the resulting solution operators into the Dirichlet-to-Neumann map.

3.1 Acoustic Scattering

Now we turn to looking at the exterior Helmholtz problem or the *scattering problem*. Say we send a sound (or electromagnetic) wave at an object and we want to see how it interacts with the object. This is the scattering problem. Appropriately, we call the object we probe with the wave a *scatterer*.

3.1.1 Problem Statement

We will limit ourselves to scatterers and waves in $\mathbb{R}^2$, but the idea generalizes to higher dimensions and we include a note on how this is carried out.

If we assume the wave speed c in the medium is constant, then the wave is described by a velocity potential $U(x, t)$ which satisfies the wave equation

$$U_{xx} = \frac{1}{c^2} U_{tt}$$

We make the additional assumption standard in classical scattering theory that the acoustic wave is a plane wave and harmonic in time

$$U(x, t) = u(x) e^{-i\omega t} \quad (3.1)$$

with $\omega > 0$. Then the space dependent component, or amplitude, $u(x)$ satisfies the Helmholtz equation (sometimes called the reduced wave equation)

$$u_{xx} + k^2 u = 0, \quad k = \frac{\omega}{c} \quad (3.2)$$

in the exterior of the curve defining the boundary of the scatterer.

In the scattering problem we denote the Dirichlet and Neumann boundary conditions by the following:

Definition 3.1.1 (Sound Soft Boundary).

$$u(x) = 0, \quad x \in \partial D$$

Definition 3.1.2 (Sound Hard Boundary).

$$\frac{\partial u}{\partial \vec{n}}(x) = 0, \quad x \in \partial D$$

We call the acoustic wave before it interacts with the scatterer the *incidence* (or *incident*) field u^{inc} , and the wave that exists as a result of interacting with ∂D the *scattered field* u^{scat} . The total field u then satisfies

$$u(x) = u^{inc}(x) + u^{scat}(x)$$

This can be equivalently expressed in polar coordinates

$$u(r, \theta) = e^{ikr \cos(\theta)}$$

In order to make the solution to the Helmholtz problem unique, we require that the scattered field satisfy the *Sommerfeld radiation condition*

Definition 3.1.3 (Sommerfeld Radiation Condition).

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^{scat}}{\partial r} - iku^{scat} \right) = 0$$

Essentially, what this says is that the wave can only be a wave going out to infinity. It cannot be a wave coming from infinity. It ensures that far enough away from D , the time dependent scattered field $U^{scat}(x, t) = u^{scat}(x) e^{-i\omega t}$ is an outgoing cylindrical wave

$$U^{scat}(x, t) \sim \frac{F(\theta)}{\sqrt{r}} e^{i(kr - \omega t)}$$

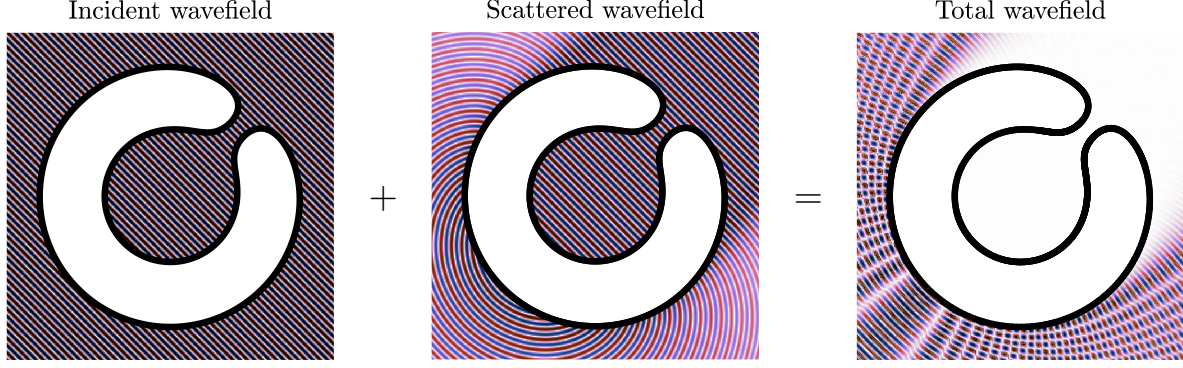


Figure 3.1: The acoustic scattering problem with sound-soft boundary conditions of a plane wave incident wavefield with $k = 20$ and $r = \frac{1}{\sqrt{2}}(1, 1)$ on the C-curve decomposed into incident wavefield (left), scattered wavefield (center) and total wavefield (right).

where $F(\theta)$ is the *far field pattern*. We will not do much with the far field pattern in this introduction but it ensures that as we go away from the scatterer at a fixed angle the amplitude decays. Note that the scattered field is the only part of the wave that must satisfy the Sommerfeld radiation condition. The incident wave need not satisfy it.

To recap, if we want to solve the acoustic scattering problem by a plane, time harmonic wave, then we solve for the scattered field u^{scat} . For convention, make the change $u(x) = u^{scat}(x)$ (then $u^{tot} = u^{inc} + u$). Then u solves

$$\begin{aligned} \Delta u + k^2 u &= 0, & x \in \mathbb{R}^2 \setminus \overline{D} \\ u(x) &= -u^{inc}, & x \in \partial D \\ &\text{or} \\ \frac{\partial u}{\partial \vec{n}}(x) &= -\frac{\partial u^{inc}}{\partial \vec{n}}(x), & x \in \partial D \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) &= 0 \end{aligned}$$

Like many problems in PDEs, if we try to solve the acoustic scattering problem on a circle or sphere, we can use separation of variables to arrive at a closed form solution. We omit the details involving Bessel functions and spherical harmonics. If the reader wishes to indulge oneself, see [2]. An example of the total wavefield when a plane wave incident wavefield interacts with the C-curve, with both sound-soft and sound-hard boundary conditions, is shown in Figure 3.1 and Figure 3.2

We are interested in the more general case that a scatterer is some curve in $\mathbb{R}^2$. How do we solve the problem here? Can it be solved at all? If so is the solution unique? The answer to the last two questions is the affirmative, though with more machinery than separation of variables.

Without loss of generality, assume our domain D contains the origin.

Remark 3.1.4. For any solution of the Helmholtz equation with Dirichlet boundary data, we would need

$$u \in C^2(\mathbb{R}^2 \setminus \overline{D}) \cap C(\mathbb{R}^2 \setminus D) \quad (3.3)$$

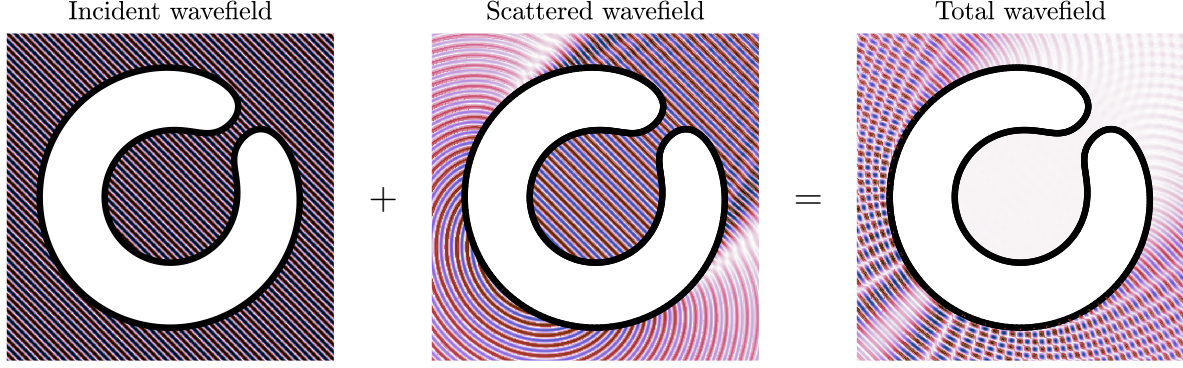


Figure 3.2: The acoustic scattering problem with sound-hard boundary conditions of a plane wave incident wavefield with $k = 20$ and $r = \frac{1}{\sqrt{2}}(1, 1)$ on the C-curve decomposed into incident wavefield (left), scattered wavefield (center) and total wavefield (right).

and for Neumann boundary data we need

$$u \in C^2(\mathbb{R}^2 \setminus \overline{D}) \cap C^1(\mathbb{R}^2 \setminus D) \quad (3.4)$$

Colton and Kress have shown that (3.3) implies (3.4). Thus we can assume any solution to the Helmholtz equation, and thus the scattering problem, satisfies (3.4).

3.1.2 Solutions on the circle and in the far-field

WV: [Levent stuff](#).

3.1.3 The Fundamental Solution

Now, like with the Laplace equation, we have a fundamental solution of the Helmholtz equation.

Theorem 3.1.5. *The (radiating) fundamental solution of the Helmholtz equation (3.2) is*

$$\phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|) \quad (3.5)$$

where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind.

Proof. This follows a similar approach as the fundamental solution to the Laplace equation. Using radial symmetry and using polar coordinates, we obtain the Hankel ODE, due to the Sommerfeld radiation condition we choose the outgoing Hankel functions. The gory details are left to the reader. $\square$

In the interest of the double-layer potential, we note that the normal derivative of the fundamental solution is given by

$$\frac{\partial \phi}{\partial \vec{n}(y)}(x, y) = \frac{ik}{4} H_1^{(1)}(k|x - y|) \frac{\vec{n}(y) \cdot (x - y)}{|x - y|}, \quad (3.6)$$

with $H_1^{(1)}$ the first order Hankel function of the first kind. We now state some facts about the fundamental solution and its normal derivatives that follow from properties of Bessel and Hankel functions.

Observation 3.1.6. The fundamental solution to the Helmholtz equation (3.5) and its normal derivative (3.6) satisfy the Sommerfeld radiation condition with respect to x and y .

Observation 3.1.7. As $|x - y| \rightarrow 0$

$$\phi(x, y) = \frac{1}{2\pi} \log \left(\frac{1}{k|x - y|} \right) + O(1), \quad (3.7)$$

and

$$\frac{\partial \phi}{\partial \vec{n}(y)}(x, y) = \frac{1}{2\pi} \frac{\vec{n}(y) \cdot (x - y)}{|x - y|^2} + O(1). \quad (3.8)$$

Observation 3.1.8. As $|x - y| \rightarrow \infty$ and $k > 0$ real,

$$\phi(x, y) \sim \sqrt{\frac{1}{8\pi k|x - y|}} e^{i(k|x - y| + \frac{\pi}{4})}, \quad (3.9)$$

and

$$\frac{\partial \phi}{\partial \vec{n}(y)}(x, y) \sim \sqrt{\frac{k}{8\pi}} \frac{\vec{n}(y) \cdot (x - y)}{|x - y|^{3/2}} e^{i(k|x - y| - \frac{\pi}{4})}. \quad (3.10)$$

The fundamental solution and its normal derivative are plotted relative to their asymptotic behavior in Figure 3.3. Now we are ready to present a boundary integral formulation of the solution to the Helmholtz equation

Theorem 3.1.9. *Let $u(x)$ be a solution of the Helmholtz equation in the exterior of D that satisfies the Sommerfeld radiation condition. Then for $x \in \mathbb{R}^2 \setminus \overline{D}$*

$$u(x) = \int_{\partial D} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u}{\partial \vec{n}}(y) \phi(x, y) \right] ds(y) \quad (3.11)$$

Proof. Let $x \in \mathbb{R}^2 \setminus \overline{D}$ and draw a disk of radius ϵ about x such that the ball is entirely in the exterior of D . That is $\Omega_{x,\epsilon} = \{y : |x - y| < \epsilon\}$ and $\Omega_{x,\epsilon} \cap \overline{D} = \emptyset$.

Draw a second disk Ω with radius R about the origin that encloses both D and $\Omega_{x,\epsilon}$.

By Green's second identity, the integral over all the boundaries of $\partial D, \partial \Omega, \partial \Omega_{x,\epsilon}$ is zero

$$\int_{\partial D + \partial \Omega + \partial \Omega_{x,\epsilon}} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) = 0$$

After the appropriate sign changes to ensure the normal derivative is always pointing outwards of the domain we enclose with the three curves we have explicitly

$$\begin{aligned} & - \int_{\partial D} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) \\ & + \int_{\partial \Omega} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) \\ & - \int_{\partial \Omega_{x,\epsilon}} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) = 0 \end{aligned}$$

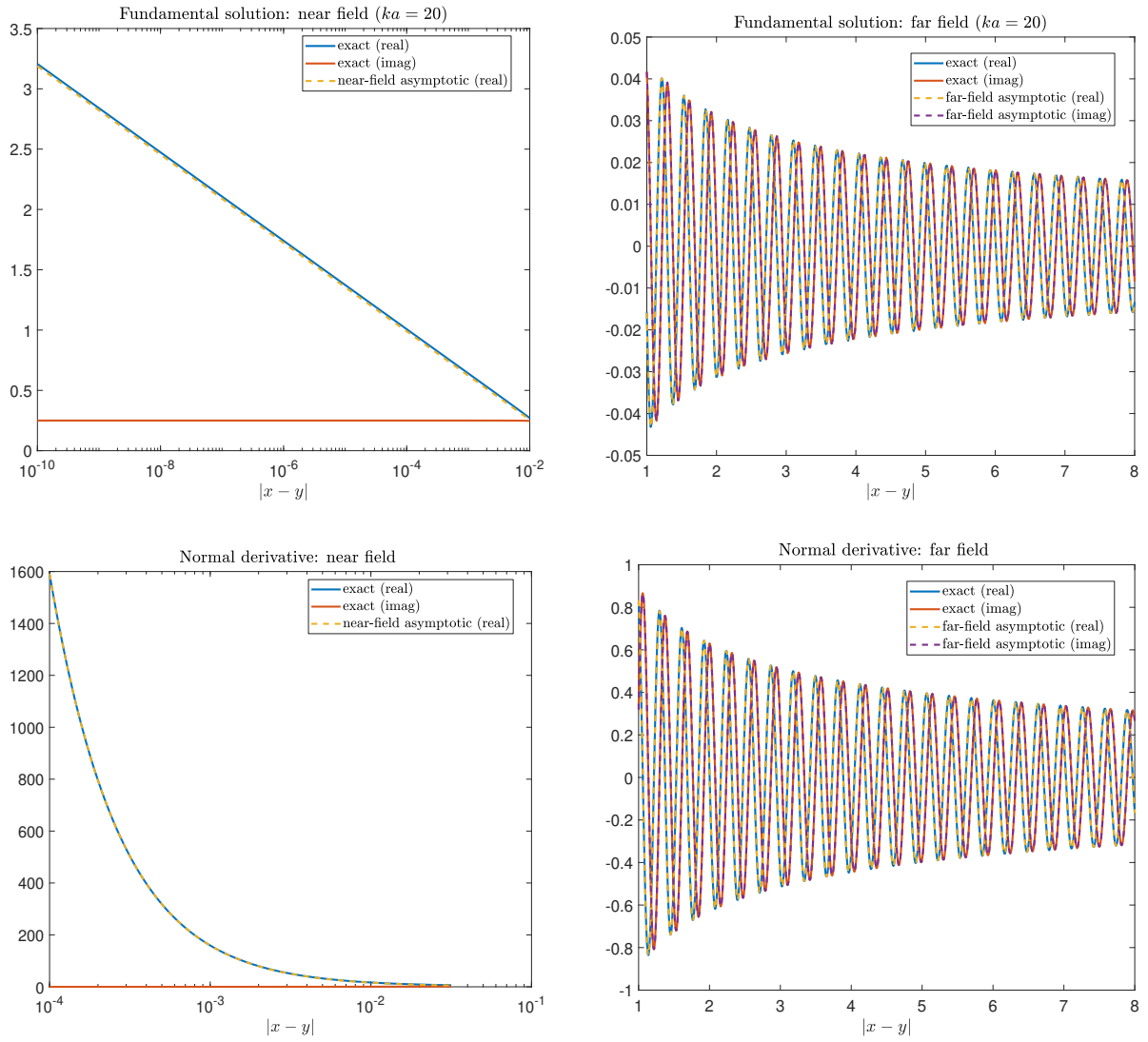


Figure 3.3: The fundamental solution to the Helmholtz equation at wave number $k = 20$. The observed asymptotic behavior is plotted over the fundamental solutions.

In the proof, we want to show that the contribution from the integrals over the boundaries of $\partial\Omega$ and $\partial\Omega_{x,\epsilon}$ amount to 0 and $-u(x)$, respectively.

On $\partial\Omega_{x,\epsilon}$ the normal derivative of (3.1.5) is

$$\frac{\partial}{\partial \vec{n}(y)} \phi(x, y) = \frac{1}{2\pi} \frac{1}{|x - y|} + O(|x - y| \log(|x - y|))$$

Taking $\epsilon \rightarrow 0$ in $\Omega_{x,\epsilon}$ means that on $\partial\Omega_{x,\epsilon}$, $|x - y| \rightarrow 0$. Further, we can express the coordinate on the boundary with $y = x + \epsilon \vec{n}$. This change of variables gives us $ds(y) = \epsilon d\vec{n}$

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_{x,\epsilon}} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) \quad (3.12)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_{x,\epsilon}} \left[u(x + \epsilon \vec{n}) \frac{\partial}{\partial \vec{n}(x + \vec{n})} \phi(x, x + \vec{n}) - \frac{\partial u(x + \vec{n})}{\partial \vec{n}(x + \vec{n})} \phi(x, x + \vec{n}) \right] \epsilon d\vec{n} \quad (3.13)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_{x,\epsilon}} \left[u(x + \epsilon \vec{n}) \frac{1}{2\pi\epsilon} - \frac{\partial u(x + \vec{n})}{\partial \vec{n}(x + \vec{n})} \frac{1}{2\pi} \log\left(\frac{1}{\epsilon}\right) \right] \epsilon d\vec{n} = u(x) \quad (3.14)$$

So we have shown that the contribution from the small circle is exactly $u(x)$. Next we use the Sommerfeld radiation condition as $R \rightarrow \infty$ to show that

$$\lim_{R \rightarrow \infty} \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) = 0$$

From these two results we get

$$\begin{aligned} & - \int_{\partial D} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) \\ & + \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) \\ & - \int_{\partial\Omega_{x,\epsilon}} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) = \\ & - u(x) + \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y) = 0 \end{aligned}$$

which gives us

$$u(x) = \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y)$$

□

From this theorem we can also make a statement about the solution to the Helmholtz equation on the interior of a domain.

Corollary 3.1.10. *Let D be any domain and $u(x)$ a solution of the Helmholtz equation in D . If $x \in D$ and $\Omega = \{y : |x - y| < \epsilon\} \subseteq D$, then*

$$u(x) = - \int_{\partial\Omega} \left[u(y) \frac{\partial}{\partial \vec{n}(y)} \phi(x, y) - \frac{\partial u(y)}{\partial \vec{n}(y)} \phi(x, y) \right] ds(y)$$

We also have that the fundamental solution $\phi(x, y)$ is an analytic function with respect to x_1, x_2 such that $x = (x_1, x_2)$ and where $x \neq y$. This gives us the following result on the analyticity of $u(x)$

Theorem 3.1.11. *Solutions $u(x)$, with $x = (x_1, x_2)$ of the Helmholtz equation are analytic functions with respect to x_1 and x_2 .*

The idea of the proof is similar to the Laplace case and rests on the analyticity of the fundamental solution away from the boundary of the domain.

3.2 Uniqueness of Solutions

We move onto the important issues of existence and uniqueness of solutions to the Helmholtz equations using the machinery of boundary integral equations.

Uniqueness is tackled first, as we will use it in establishing existence. We stress that uniqueness means a single solution. First, we prove a special case of Holmgren's Theorem.

Theorem 3.2.1. *Let $u(x)$ be a solution of the Helmholtz equation in the exterior of a domain D satisfying the Sommerfeld radiation condition and with zero boundary data. Then $u(x) \equiv 0$ for $x \in \mathbb{R}^2 \setminus \overline{D}$*

Proof. We present an outline of the proof. Let Ω be a disk centered at the origin containing D in its interior. We know that $u(x)$ is analytic in $\mathbb{R}^2 \setminus \overline{D}$. Thus in $\mathbb{R}^2 \setminus \Omega$, u has a series representation

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta}$$

$$a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r, \theta) e^{-in\theta} d\theta$$

Using asymptotic properties the Bessel and Hankel functions, we show that

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = 0$$

which means each coefficient a_n is zero. This implies $u(x) \equiv 0$ on $\mathbb{R}^2 \setminus \Omega$ which in turn implies $u(x) \equiv 0$ on $\mathbb{R}^2 \setminus \overline{D}$ by the analyticity of solutions to the Helmholtz equation. $\square$

A uniqueness proof follows immediately.

Theorem 3.2.2. *If a solution to the Helmholtz equation exists, then it is unique.*

The proof is a standard uniqueness proof in analysis.

Proof. Assume $u_1(x)$ and $u_2(x)$ are both solutions to the Helmholtz equation given some boundary data. Consider the difference equation $u(x) = u_1(x) - u_2(x)$.

On ∂D , $u(x) \equiv 0$. From Theorem 3.2.1 we have $u(x) \equiv 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Thus $u_1 = u_2$ in all of $\mathbb{R}^2 \setminus \overline{D}$ and we are finished. $\square$

WV: Lots should be added here. Colton and Kress prove in Integral equation methods in scattering theory chapter 3 that for $k \in \mathbb{C} \setminus \mathbb{R}$, the solution has at most one solution on the interior domain. Then you can also prove uniqueness on the exterior domain for complex wave numbers in the same book. This is quite crucial. Existence for complex wave numbers is a tougher cookie to crack. The Laplacian eigenvalue problem on the exterior domain can indeed not be solved for real wave numbers, but that's not what we are trying to solve, we have the Sommerfeld radiation condition! The Laplacian eigenvalue problem is not solvable for real k for solutions in L^2 , our solutions that we want to find for this problem is not in L^2 ! We should look into the limiting absorption principle, the solution we are looking for is the limit of the damped Helmholtz equation as it loses dampening. Now for the BIE we are not looking at any limit, so why does it exist? The operator we are inverting goes from boundary data (Dir or Neu) to boundary data (layer function). The BIE encodes the PDE, but also the radiation condition in the operator, making it invertible, then it's all just the Fredholm alternative here. Now for the complex wave numbers, it's again radiation condition choice. We chose for outgoing waves, the wave numbers related to negative imaginary part are associated with outgoing wave exploding. Choosing the incoming waves instead makes the upper half plane a problem. The affect of the Sommerfeld radiation condition written down would be insanely helpful as I haven't found a places where I like the explanation of why it works.

BP: Add paragraph on what we did in Laplace, how it still makes sense, and how the Sommerfeld radiation condition gives us existence.

3.3 Existence of Solutions

With uniqueness squared away we move towards existence. We do so through boundary integral equations and layer potentials. The reader will notice that the arguments presented are similar in spirit to those for the existence of Laplace's equation. Indeed, much is the same between the treatment of the two equations with boundary integral equations, due to their ellipticity.

BP: I've heard this alluded to but would need a reference to feel okay without a caveat note.

WV: It'd be amazing if we could find a results that proves that the Helmholtz solution in the exterior domain exists for $\text{Im}(k) \geq 0$.

3.3.1 Layer Potentials

Recall the fundamental solution to the Helmholtz equation from Def 3.1.5

$$\phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|)$$

Like in the case of the Laplace equation, we use the fundamental solution to define the single and double layer operators.

Definition 3.3.1 (Single- and double-layer operators). Given a continuous density function τ defined on the boundary ∂D of a domain D , the integral operators

$$\mathcal{S}\tau = \int_{\partial D} s(\cdot, y)\tau(y) ds(y) \tag{3.15}$$

and

$$\mathcal{D}\tau = \int_{\partial D} d(\cdot, y)\tau(y) ds(y) \tag{3.16}$$

The single- and double-layer kernel on a curve

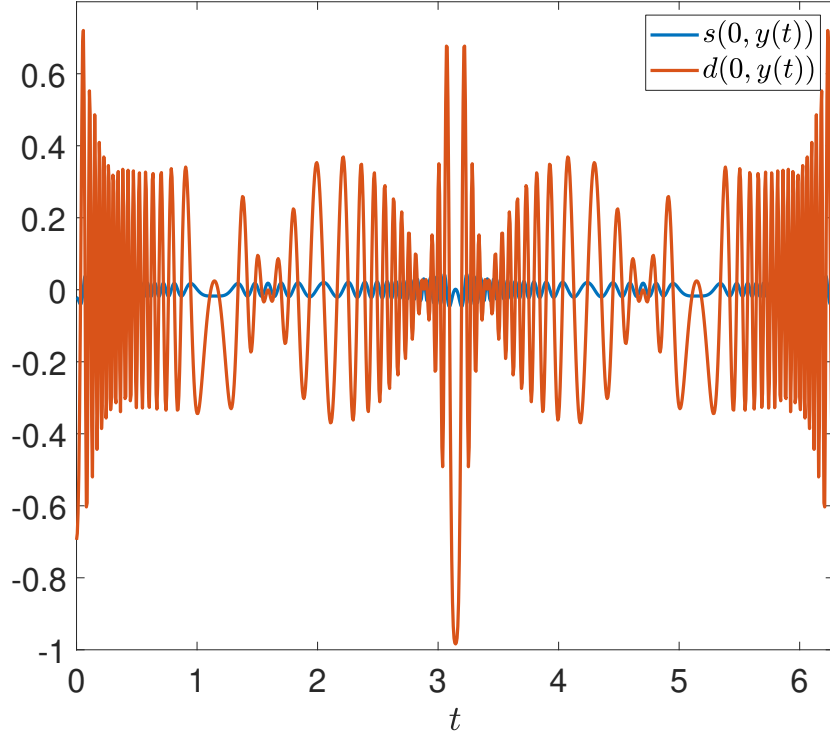


Figure 3.4: The real part of the single- and double-layer kernel for the Helmholtz equation at target point $x = 0$ and wave number $k = 20$ on the C-curve.

are called the *single-layer operator* and *double-layer operator*, respectively. The single and double layer kernels are defined as the fundamental solution and the normal derivative of the fundamental solution, respectively.

$$s(x, y) = \phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|)$$

and

$$d(x, y) = \frac{\partial \phi}{\partial \vec{n}(y)} = -\frac{ik}{4} H_1^{(1)}(k|x - y|) \frac{(x - y) \cdot \vec{n}(y)}{|x - y|}$$

The double- and single-layer kernel for the Helmholtz equation are shown in Figure 3.4. As was the case with the Laplace equation, we can use these operators to define the layer potentials.

Definition 3.3.2 (Single- and double-layer potentials). Given a continuous density function τ defined on the boundary ∂D of a domain D , we call

$$u(x) = [\mathcal{S}\tau](x) = \int_{\partial D} s(x, y) \tau(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D \quad (3.17)$$

the *acoustic single-layer potential*. Analogously, we call

$$v(x) = [\mathcal{D}\tau](x) = \int_{\partial D} d(x, y) \tau(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D \quad (3.18)$$

the *acoustic double-layer potential*.

Through the single and double-layer potentials, we can express the solution to the Helmholtz equation for interior and exterior problems (i.e. scattering problems) as an integral equation evaluated only on the boundary of the scatterer.

3.3.2 Jump Conditions

In order to define the boundary integral operators and equations we will solve to find solutions to the Helmholtz equation we need to understand how the layer potentials behave as we approach ∂D from the interior and exterior of the domain. As was the case in Section 2.4.2, the single and double layer kernels for the Helmholtz equation are weakly singular. Thus, the operators $\mathcal{S}$ and $\mathcal{D}$ are compact operators! Further, the jump conditions are the same as the Laplace case.

Explicitly, we can extend the double-layer potential from $\mathbb{R}^2 \setminus \overline{D}$ to ∂D .

Theorem 3.3.3 (Double-layer potential jump condition). *Let $u(x)$ be the double-layer potential with continuous density τ on a domain D of class C^2 . The potential can be extended to ∂D such that*

$$v_{\pm}(x) = \pm \frac{1}{2}\tau(x) + \int_{\partial D} \frac{\partial \phi(x, y)}{\partial \vec{n}(y)} \tau(y) ds(y), \quad x \in \partial D \quad (3.19)$$

where the sign $+$ corresponds to approaching the boundary from the exterior of D and $-$ corresponds to approaching the boundary from the interior.

Proof. The proof follows the same splitting strategy as the proof of the double-layer potential jump condition for the Laplace equation in Theorem 2.4.4. Indeed we can see that the singularities and their asymptotic behaviors (3.7) and (3.8) are the same as in the double-layer potential of the Laplace equation. The rest of the proof is filled in much the same as in Theorem 2.4.4. $\square$

The single-layer potential does not emit a jump condition. It extends unaltered from $\mathbb{R}^2 \setminus \overline{D}$ to ∂D .

Theorem 3.3.4. *Let $u(x)$ denote the single-layer potential defined on $\mathbb{R}^2 \setminus \overline{D}$. The potential can be continuously extended to ∂D such that*

$$u(x) = \int_{\partial D} s(x, y) \tau(y) ds(y), \quad x \in \partial D$$

where the integral exists in the improper sense.

3.3.3 Solution Representations

Using the single and double-layer potentials defined on ∂D , we can express solutions to the exterior Dirichlet and Neumann Helmholtz problems. With a focus on the scattering problem, we use the incident field data for the boundary data and require that the solutions satisfy the Sommerfeld radiation condition (3.1.3).

We state these solution representations in the following two theorems.

Theorem 3.3.5. *The acoustic double-layer potential solves the exterior Dirichlet problem*

$$\begin{cases} \Delta u + k^2 u = 0, & x \in \mathbb{R}^2 \setminus \overline{D} \\ u(x) = -u^{inc}(x), & x \in \partial D \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \end{cases}$$

if the density layer τ is a solution of the integral equation

$$\frac{1}{2}\tau(x) + \int_{\partial D} d(x, y)\tau(y) ds(y) = -u^{inc}(x), \quad x \in \partial D$$

In operator notation, this integral equation is given by $(\frac{1}{2}I + \mathcal{D})\tau = -u^{inc}$. The operator $\frac{1}{2}I + \mathcal{D}$ is a boundary integral operator.

Theorem 3.3.6. *The acoustic single-layer potential solves the exterior Neumann problem*

$$\begin{cases} \Delta u + k^2 u = 0, & x \in \mathbb{R}^2 \setminus \overline{D} \\ \frac{\partial u(x)}{\partial x} = -\frac{\partial u^{inc}(x)}{\partial x}, & x \in \partial D \\ \lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0 \end{cases}$$

if the density layer τ is a solution of the integral equation

$$-\frac{1}{2}\tau(x) + \int_{\partial D} \frac{\partial \phi(x, y)}{\partial \vec{n}(x)} \tau(y) ds(y) = -\frac{\partial u^{inc}(x)}{\partial \vec{n}(x)}$$

In operator notation, this integral equation is given by $(-\frac{1}{2}I + \mathcal{D}')\tau = -\frac{\partial u^{inc}}{\partial \vec{n}(\cdot)}$. The operator $-\frac{1}{2}I + \mathcal{D}'$ is a boundary integral operator.

3.3.4 Fredholm Alternative

These boundary integral operators are of the kind with which we can apply the Fredholm Alternative. In order to do so, we need to understand the nullspaces of the aforementioned boundary integral operators. We will see that these operators have nullspaces which depend on the parameter k , and indeed they emit problematic nullspaces at certain values of k .

Before making a statement on the nullspaces of the boundary integral operators, we will suggestively define

$$U := \left\{ u(x), x \in \partial D : \Delta u + k^2 u = 0 \text{ in } D, \text{ and } \frac{\partial u(x)}{\partial \vec{n}(x)} = 0 \text{ for } x \in \partial D \right\} \quad (3.20)$$

$$V := \left\{ \frac{\partial u(x)}{\partial \vec{n}(x)}, x \in \partial D : \Delta v + k^2 v = 0 \text{ in } D, \text{ and } v(x) = 0 \text{ for } x \in \partial D \right\} \quad (3.21)$$

where $u, v \in C^1(\overline{D})$.

Recall that the nullspace of $\frac{1}{2}I + \mathcal{D}$ corresponds to the eigenspace of $\mathcal{D}$ since

$$\left(\frac{1}{2}I + \mathcal{D} \right) \tau = 0 \implies \mathcal{D}\tau = -\frac{1}{2}\tau$$

which means exactly that τ is an eigenfunction of the operator $\mathcal{D}$. An analogous statement can be made for $\frac{1}{2}I - \mathcal{D}'$

This eigenspace-nullspace connection can also be seen in the definitions of (3.20) and (3.21). Consider (3.20). If $u \in U$, then $\Delta u = -k^2 u$ in D subject to the Neumann boundary conditions. Then if k^2 is not an eigenvalue of the interior Neumann problem, the only u which can satisfy the requirements is $u \equiv 0$, i.e. $U = \{0\}$. Alternatively, this means k^2 is an eigenvalue of the negative Laplace operator defined on the interior of the scatterer. Similarly, if k^2 is not an eigenvalue of the interior Dirichlet problem, then $V = \{0\}$.

The reader might be able to guess the statement we have been building up to. We make the reveal now.

Theorem 3.3.7. *The nullspace of $\frac{1}{2}I + \mathcal{D}$, $\mathbf{N}(\frac{1}{2}I + \mathcal{D})$ is given by U .*

Proof. Assume $\tau \in \mathbf{N}(\frac{1}{2}I + \mathcal{D})$. Take $u(x)$ to be the double-layer potential in $\mathbb{R}^2 \setminus \partial D$. Define $u_+(x)$ as the limit of $u(x)$ to ∂D from the exterior of D .

$$u_+(x) = \lim_{x \rightarrow \partial D} u(x), \quad x \in \mathbb{R}^2 \setminus \overline{D}$$

Then $u_+(x) = 0$. From Theorem 3.2.1 we have that $u(x) \equiv 0$ for $x \in \mathbb{R}^2 \setminus D$. Since we have $u(x) \equiv 0$ outside the scatterer, the normal derivative is zero as we limit to the boundary from the exterior. That is

$$\frac{\partial u_+}{\partial \vec{n}(x)} = 0$$

The normal derivative of the double layer potential is continuous across the boundary ∂D which gives us

$$\frac{\partial u_+}{\partial \vec{n}(x)} = 0 = \frac{\partial u_-}{\partial \vec{n}(x)}$$

Since we have zero valued Neumann boundary data from the interior of the scatterer, and the double-layer potential is valid for all $x \in D$, we have that $u(x)$ solves the interior Neumann problem. The difference of the double-layer potential across ∂D is exactly the density layer. Combining this with the zero value data in the exterior of D , we have

$$\tau = u_+ - u_- = -u_-$$

Writing out the chain explicitly and using that u_- is the double layer potential, we have

$$\begin{aligned} -u_- &= \tau \\ \tau + u_- &= 0 \\ \tau + \int_{\partial D} \frac{\partial \phi(x, y)}{\partial \vec{n}(y)} \tau(y) ds(y) &= 0 \\ (I + \mathcal{D})\tau &= 0 \end{aligned}$$

which means $\tau \in U$.

Now to show that $\tau \in U \implies \tau \in \mathbf{N}(I + \mathcal{D})$. Assume $\tau \in U$. Green's second identity gives us

$$\int_{\partial D} \frac{\partial \phi(x, y)}{\partial \vec{n}(y)} \tau(y) ds(y) = 0$$

Taking $x \rightarrow \partial D$ we get the boundary integral

$$\tau + \mathcal{D}\tau = 0$$

which means $\tau \in \mathbf{N}(I + \mathcal{D})$ □

We present the following result on the nullspace of the operator $-\frac{1}{2}I + \mathcal{D}'$.

Theorem 3.3.8. *The nullspace of $-\frac{1}{2}I + \mathcal{D}'$, $\mathbf{N}(-\frac{1}{2}I + \mathcal{D}')$ is given by V .*

Proof. Assume $\tau \in \mathbf{N}(-\frac{1}{2}I + \mathcal{D}')$. Let $u(x)$ be the single-layer potential in $\mathbb{R}^2 \setminus \partial D$. Define

$$\frac{\partial u_+(x)}{\partial \vec{n}(x)} = \lim_{x \rightarrow \partial D} \frac{\partial u(x)}{\partial \vec{n}(x)}, \quad x \in \mathbb{R}^2 \setminus \overline{D}$$

From the assumption we have $\frac{\partial u_+(x)}{\partial \vec{n}(x)} = 0$. The uniqueness of the exterior Neumann problem gives us $u \equiv 0$ for $x \in \mathbb{R}^2 \setminus \overline{D}$. This implies $\frac{\partial u_+(x)}{\partial \vec{n}(x)} = 0$. The normal derivative of the single-layer potential is continuous across the boundary ∂D , thus $\frac{\partial u_+(x)}{\partial \vec{n}(x)} = 0 = \frac{\partial u_-(x)}{\partial \vec{n}(x)}$. From here we see that u solves the interior Dirichlet problem, thus $\tau \in V$.

For the reverse, assume $\tau \in V$. Green's first theorem gives us

$$v(x) = \int_{\partial D} \left[\frac{\partial v(y)}{\partial \vec{n}(x)} \phi(x, y) + \frac{\partial \phi(x, y)}{\partial \vec{n}(x)} v(y) \right] ds(y) = \int_{\partial D} \phi(x, y) \tau(y) ds(y)$$

So $v(x)$ is the single-layer potential on the boundary. Then $v(x) = u_-(x)$. Limiting to the boundary, we have

$$\tau = \frac{\partial u_-(x)}{\partial \vec{n}(x)} = \left(\frac{1}{2}I + \mathcal{D}' \right) \tau$$

With some algebra we see

$$\left(\frac{1}{2}I + \mathcal{D}' \right) \tau = \tau \implies -\frac{1}{2}\tau + \mathcal{D}'\tau = 0$$

Thus $\tau \in \mathbf{N}(-\frac{1}{2}I + \mathcal{D}')$. □

Finally, with the nullspaces defined for the boundary integral operators which the density layers must satisfy to solve an exterior problem, we can use the Fredholm alternative to state the unique existence of solutions.

Theorem 3.3.9 (Unique Existence of Solutions to Exterior Dirichlet Problem). *The exterior Dirichlet problem as stated in Theorem 3.3.5 is uniquely solvable when k is not an eigenvalue of the interior Neumann problem.*

Proof. Follows immediately from the Fredholm alternative and Theorem 3.3.7. □

Theorem 3.3.10 (Unique Existence of Solutions to Exterior Neumann Problem). *The exterior Neumann problem as stated in Theorem 3.3.5 is uniquely solvable when k is not an eigenvalue of the interior Dirichlet problem.*

Proof. Follows immediately from the Fredholm alternative and Theorem 3.3.8. □

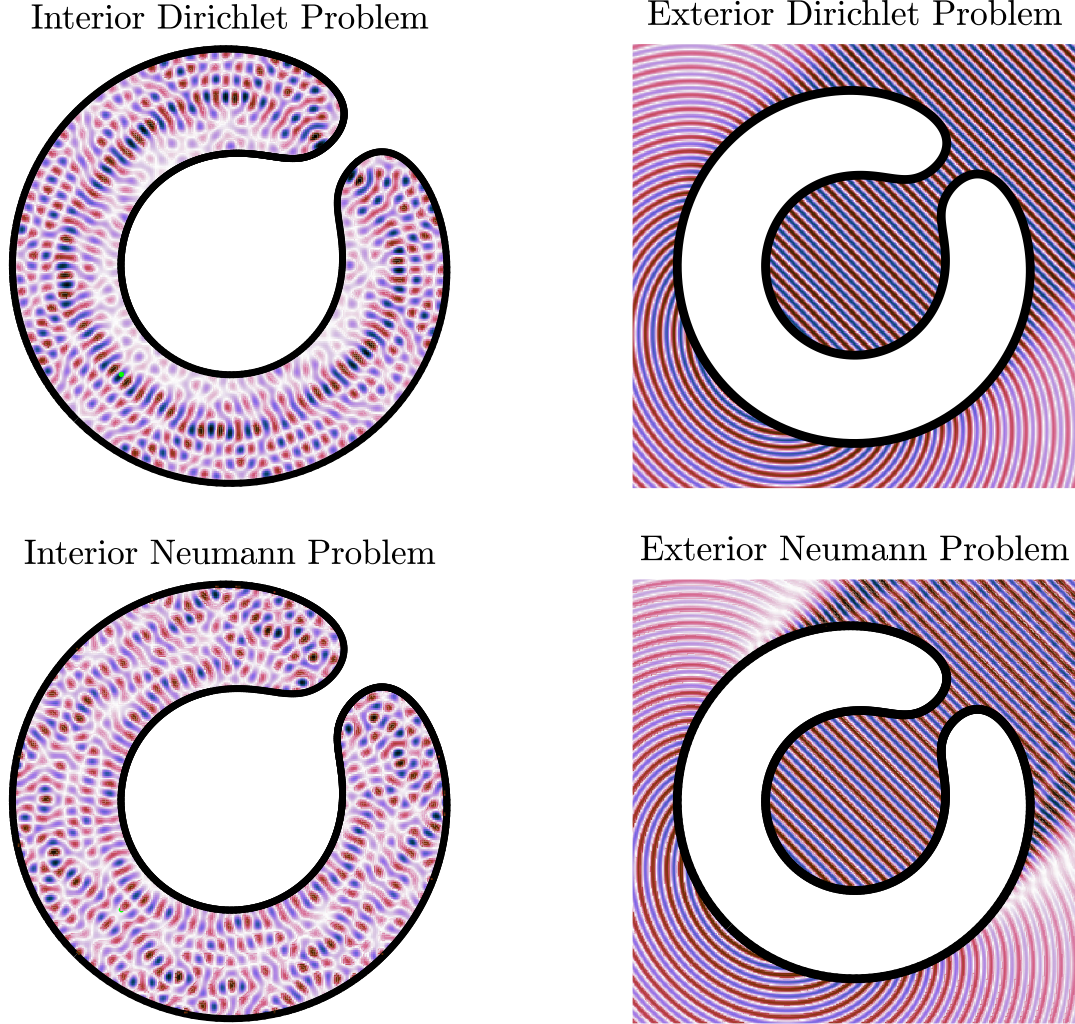


Figure 3.5: The real part of solutions to Helmholtz boundary value problems at wave number $k = 20$. The exterior problems has a plane wave imposed at frequency $k = 20$ with direction $r = \frac{1}{\sqrt{2}}(1, 1)$ and the interior problems have boundary conditions arriving from a point source inside the domain.

Normally, proving unique existence should leave us satisfied, but these come with a caveat. If we happen to try to solve the exterior problem at an eigenvalue of an interior problem, our boundary integral machinery breaks down, and we are left stranded. Indeed, for a general scatterer we will not know where these interior eigenvalues are, and so solving the exterior problem comes down to a matter of luck. Numerically, the situation is even worse. Instead of only avoiding a problematic wave number, we have to avoid a neighborhood of it. But not all hope is lost! We will see in a subsequent section how to remedy this issue and arrive at a boundary integral formulation that is valid for a much more inclusive set of wave numbers.

WV: We should add a short description of existence for the single layer potential and then show a figure where we plot the norm of the solution at some points and show that it does in fact blow up at the eigenvalues where we say it should blow up. This also shows the comment on "avoid a neighborhood of it."

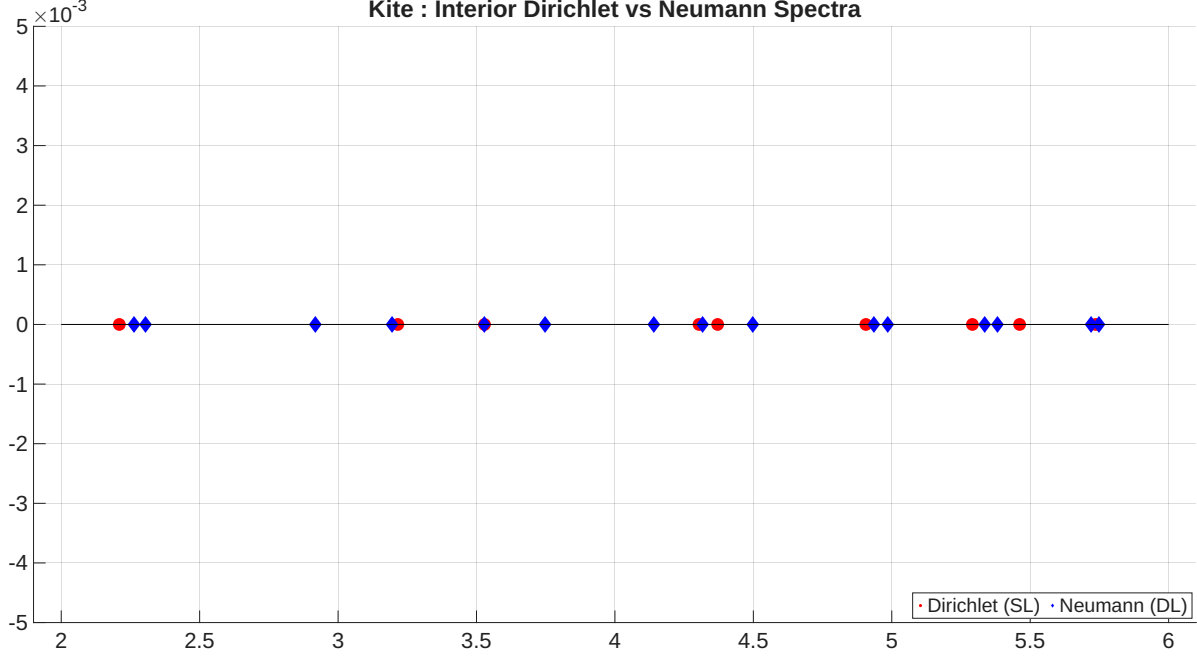


Figure 3.6: The points where the boundary integral formulation of the exterior Dirichlet problem (red circles) and Neumann problem (blue diamonds) fail to be well-posed for a kite scatterer Figure 3.7. These correspond to the spectra of the single and double layer boundary integral operators, respectively.

3.4 The Combined Field Integral Equations

Boundary integral equations for the exterior problems using double- and single-layer operators separately have now been shown to be ill-posed problems when the wave number is an eigenvalue of an interior problem. However, the spectra of the two interior problems seem to be disjoint. While the solution to the Helmholtz equation might not be well-posed through the single-layer potential approach at a certain wave number, the double-layer potential approach will yield a solution. Finding one BIE equation that is well-posed for all real wave numbers rigorously proves the existence of the solution. In a similar vain to Theorem 2.4.6, the Brakhage-Werner Leis, Panich's approach is to use a linear combination of the double- and single-layer potential gives a solution approach to the Helmholtz equation that is well-posed for all real wave numbers [3]. This combination is called the combined field integral formulation. The resulting combined field integral equation (CFIE) is used for exterior problems since we know that the scattering problem observes no resonant modes on the real line, on interior problems, the resonant modes are on the real line and so a BIE where the resonant modes aligns with its ill-posed wave numbers should be chosen. We see this in Figure 3.6.

3.4.1 The sound-soft problem

We state and prove the CFIE for the exterior Dirichlet problem with boundary condition [3, 8].

Theorem 3.4.1. *Let D , be a bounded domain of class C^2 and $D^C = \mathbb{R}^m \setminus \overline{D}$,*

$$u(x) = [(\mathcal{D} - i\eta\mathcal{S})\tau](x), \quad x \in D^C, \quad (3.22)$$

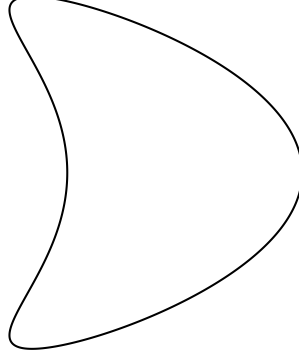


Figure 3.7: The kite scatterer used for the operator spectra figures in this section. It is closed, smooth, and features no trapping features

with $\eta \in \mathbb{R}$ a non-zero coupling parameter and density function $\tau \in C$ s.t.

$$\frac{\tau(x)}{2} + [(\mathcal{D} - i\eta\mathcal{S})\tau](x) = f(x), \quad (3.23)$$

is a solution to the exterior Dirichlet problem with boundary condition f .

The proof follows from the jump condition of the double-layer potential. We now show that for any non-zero coupling parameter η , the Fredholm integral equation of the second kind (3.23) is injective and through the Fredholm alternative, the inverse exists. While the proof can seem repetitive, it shows where the CFIE fails in the complex plane and indicates a choice of coupling parameter.

Theorem 3.4.2. *Let $\mathcal{S}$ and $\mathcal{D}$ be the single- and double-layer operator, then*

$$\mathcal{N}\left(\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S}\right) = \{0\},$$

for any $\eta \neq 0$ and real wave number k .

Proof. Let $\tau \in C$ such that $(\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S})\tau \equiv 0$. That mean that $u = (\mathcal{D} - i\eta\mathcal{S})\tau$ solve the exterior Helmholtz problem with Dirichlet boundary condition $u_+ \equiv 0$. Due to the uniqueness of the solution, we have that $u \equiv 0$ in D^C and therefore $\frac{\partial u_+}{\partial \vec{n}} = 0$. Now using the jump condition from the double-layer potential (2.7) and normal derivative of the single-layer potential (2.12) (and lack of jumps), we can relate the limiting behavior from the interior u_- and exterior u_+ , of the domain D ,

$$\begin{aligned} u_+ - u_- &= \tau && \text{on } \partial D \\ \frac{\partial u_+}{\partial \vec{n}} - \frac{\partial u_-}{\partial \vec{n}} &= i\eta\tau && \text{on } \partial D \end{aligned}$$

Therefore,

$$\frac{\partial u_-}{\partial \vec{n}} - i\eta u_- = 0.$$

Thus, u satisfies the Helmholtz equation everywhere but on ∂D and on the interior of the domain D has boundary condition $\frac{\partial u_-}{\partial \vec{n}} = i\eta u_-$. We have that u satisfies the Helmholtz BVP

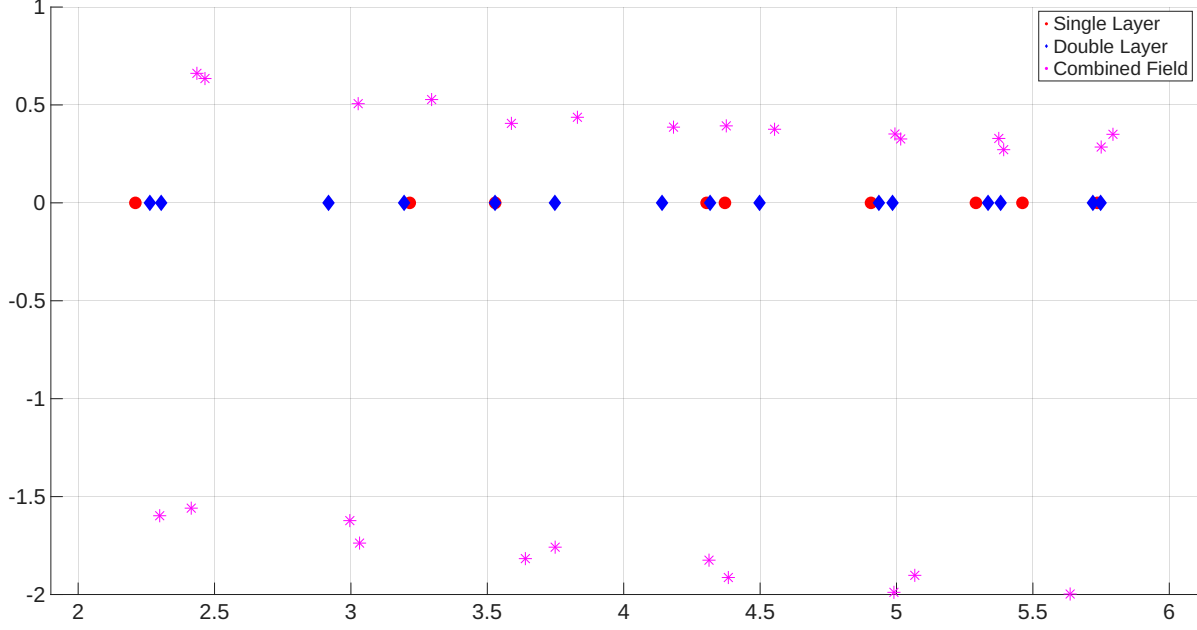


Figure 3.8: The spectra of the single layer potential (red circles), double layer potential (blue diamonds), and combined field boundary operator (purple stars) of a kite scatterer in Figure 3.7. The coupling parameter is set to -1 so we see that there exist spurious modes in the upper half plane. Notice no eigenvalues of the combined field boundary operator exist on the real axis.

with impedance boundary conditions.

$$\begin{aligned} -\Delta u_-(x) &= k^2 u_-(x) & x \in \Omega, \\ \frac{\partial u_-}{\partial \vec{n}} - i\eta u_- &= 0 & x \in \partial\Omega. \end{aligned}$$

This is exactly the eigenvalue problem of the Laplacian with impedance boundary condition seen in Theorem 2.5.9, with $\lambda = k^2$. The eigenvalues depend on the sign of η and no eigenvalues are on the real line! Therefore, when k^2 is not an eigenvalue of the Laplacian with impedance boundary conditions, the solution u is uniquely $u \equiv 0$ in D . Since k^2 is real, it cannot be an eigenvalue, hence $u \equiv 0$, more importantly $u_- = 0$ on ∂D , thus $\tau = 0$. Therefore

$$\mathcal{N}\left(\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S}\right) = \{0\}.$$

□

We now know that the CFIE is well-posed for all real wave numbers! However, there are still other wave numbers for which the CFIE is ill-posed while the Helmholtz solution still exists. It is important to know approximately where the wave numbers with ill-posed problems are located [1, 3]. As seen in observation 2.5.10, the sign of the coupling parameter η determines in which half of the complex k -plane the CFIE is ill-posed. These wave numbers are called spurious modes. They are not wave numbers where the solution to the Helmholtz equation doesn't exist, they are wave numbers where the BIE is ill-posed. Since we know the spurious nodes are only encountered for $\text{Re}(k) > 0$, since $\text{Im}(k^2) = 2\text{Re}(k)\text{Im}(k)$, we know that when

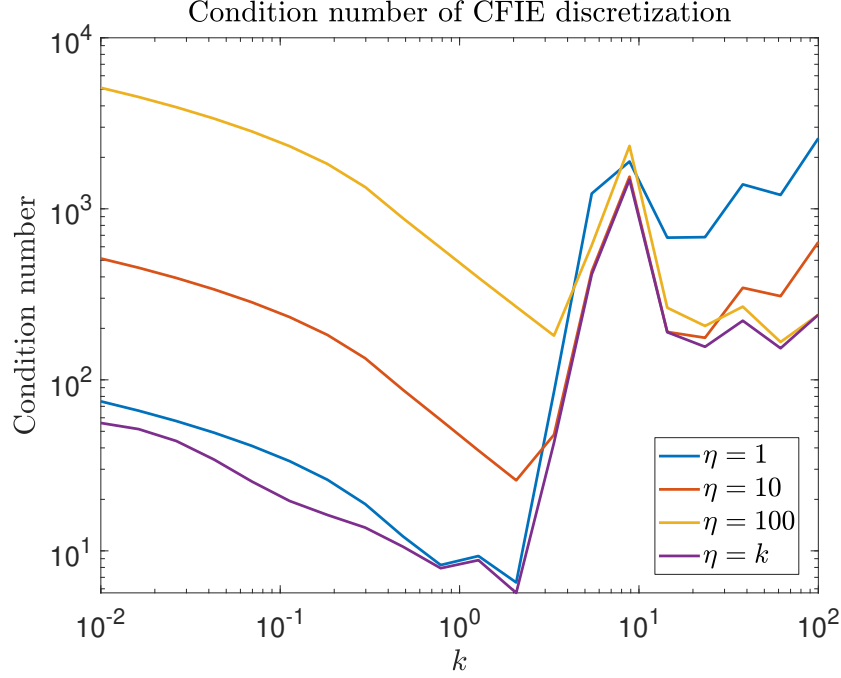


Figure 3.9: The condition number of the discretized CFIE for different coupling parameters over various wave numbers.

η is chosen to be positive the spurious modes will be in the lower half plane, while when η is negative, the spurious modes will be in the upper half plane. The distance between the spurious nodes and the real line is determined by the value of η . Through (2.21) and (2.22), we see that

$$\operatorname{Re}(k) = \sqrt{\frac{\int_{\Omega} |u|^2 \, d\Omega}{\int_{\Omega} |\nabla u|^2 \, d\Omega} + \operatorname{Im}(k)^2} \text{ and } |\operatorname{Re}(k)\operatorname{Im}(k)| \sim \eta,$$

therefore

$$|\operatorname{Im}(k)| \sim \sqrt{\eta} \text{ and } |\operatorname{Re}(k)| \sim \sqrt{\eta} \Rightarrow |k| \sim \sqrt{\eta}.$$

Try and show this happening in a figure For larger η , the spurious modes are pushed further into the complex plane. It is seen that $\eta > 1$ is a safe choice, however, we can also take the coupling parameter to be dependent on the wave number for conditioning. Indeed, let η be constant, as the wave number k grows we see, using Theorem 3.1.8,

$$\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S} \sim \frac{I}{2} + \sqrt{k} - \frac{\eta}{\sqrt{k}} \sim \frac{I}{2} + \sqrt{k} \sim \frac{I}{2} + \mathcal{D}.$$

Therefore, if η is constant, as k grows, we see that we return to the double layer potential which causes the spurious wave number to approach the real line. This leads to a commonly chosen coupling parameter $\eta = \operatorname{Re}(k)$ [1, 3], we then have that both the single and double layer terms are on the same order of magnitude relative to the wave number. However, as k grows even further, the CFIE loses the Fredholm integral equation of the second kind's conditioning in numerical approaches. The so-called high-frequency Helmholtz equation comes with many more problems, and is a topic out of the scope of this report [1, 9]. The condition number for various coupling parameters is shown in Figure 3.9.

3.4.2 The sound-hard problem

To solve the sound-hard problem, we define the adjoint to the double layer operator and the hypersingular boundary integral operator, respectively,

$$\mathcal{D}'\tau = \frac{\partial}{\partial \vec{n}(\cdot)} \mathcal{S}\tau = \int_{\partial D} \frac{\partial}{\partial \vec{n}(\cdot)} \phi(\cdot, y) \tau(y) ds(y),$$

and

$$\mathcal{T}\tau = \frac{\partial}{\partial \vec{n}(\cdot)} \mathcal{D}\tau = \frac{\partial}{\partial \vec{n}(\cdot)} \int_{\partial D} \frac{\partial}{\partial \vec{n}(y)} \phi(\cdot, y) \tau(y) ds(y).$$

Let the solution be described as

$$u = (\mathcal{D} - i\eta\mathcal{S})\tau, \quad \text{in } D^C,$$

if the density layer function satisfies

$$\left(\frac{i\eta}{2} I + \mathcal{T} - i\eta\mathcal{D}' \right) \tau = g, \quad \text{on } \partial D,$$

then u is a solution to the sound-hard scattering problem with boundary condition g . Here the jump condition followed from the normal derivative of the single layer potential and it should be noted that the jump of the double-layer potential does not survive the normal derivative, as discussed in the Laplace chapter. Note that $\mathcal{T}$ is hypersingular and therefore not compact on $C(\partial D)$, so Theorem 1.4.18 does not apply directly. Extending the layer operators to $L^2(\partial D)$ via Theorem 1.4.13 and splitting of the Laplace hypersingular part returns a compact perturbation of an invertible operator. The proof that the integral operator $\frac{i\eta}{2} I + \mathcal{T} - i\eta\mathcal{D}'$ is invertible is then nearly identical to the proof in Theorem 3.4.2 [3], relying on the uniqueness of the exterior Neumann problem instead of the exterior Dirichlet problem. The BIE is ill-posed at exactly the same spurious modes as the sound-soft problem.

3.4.3 Dirichlet-to-Neumann maps

Being able to solve the sound-soft and sound-hard problem for all real wave numbers k allows us to describe the behavior at the boundary through a density layer function. This opens up the world of Dirichlet-to-Neumann (DtN) and Neumann-to-Dirichlet (NtD) maps. These maps translates Dirichlet data at the boundary to Neumann data and Neumann data to Dirichlet data, respectively. The DtN map is given by

$$\mathcal{A}_{DtN} : C^{1,\alpha}(\partial D) \rightarrow C^{0,\alpha}(\partial D) : f \mapsto \underbrace{\left(\frac{i\eta}{2} I + \mathcal{T} - i\eta\mathcal{D}' \right)}_{\text{evaluates at boundary}} \underbrace{\left(\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S} \right)^{-1}}_{\text{solves for } \tau} f,$$

the NtD map is given by

$$\mathcal{A}_{NtD} : C^{0,\alpha}(\partial D) \rightarrow C^{1,\alpha}(\partial D) : g \mapsto \underbrace{\left(\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S} \right)}_{\text{evaluates at boundary}} \underbrace{\left(\frac{i\eta}{2} I + \mathcal{T} - i\eta\mathcal{D}' \right)^{-1}}_{\text{solves for } \tau} g.$$

Note that $\mathcal{A}_{NtD} = \mathcal{A}_{DtN}^{-1}$, they are not independently-constructed map. The mapping properties stated above are the Hölder ones inherited from the layer operators on $C(\partial D)$, but the

two applications discussed below both treat the DtN map as an operator on energy spaces rather than on Hölder spaces. Both boundary integral operators appearing in $\mathcal{A}_{DtN}$ have a transposed-kernel partner, obtained by exchanging source and target in the kernel, so Theorem 1.4.13 applies and each is bounded with respect to the $L^2(\partial D)$ norm. Since $C(\partial D)$ is dense in $L^2(\partial D)$, they extend uniquely to bounded operators there, and by Theorem 3.4.2 the extension of $\frac{I}{2} + \mathcal{D} - i\eta\mathcal{S}$ extends to an injective bounded operator for real k . The DtN map is therefore a bounded operator on $L^2(\partial D)$, and it is in this setting that the following two interpretations live. Note here that again the choice of η is related to the invertibility of the operators. In the end, the physical DtN and NtD maps are independent of the choice in η on the real line.

Beyond their role in scattering theory, DtN maps are the standard device for turning an unbounded exterior problem into a bounded one. If we want to solve a Helmholtz problem on all of $\mathbb{R}^m$ numerically, we cannot mesh an infinite domain. Instead we truncate to a bounded region containing all inhomogeneities, and impose $\mathcal{A}_{DtN}$ as an exact boundary condition on the truncation boundary. This replaces the usual approximate absorbing boundary conditions (Sommerfeld radiation truncated to low order, PML, etc.) with something exact, at the cost of the DtN map itself being nonlocal and expensive to apply [9]. This is the same idea used in non-overlapping domain decomposition methods, where the Schur complement appearing after eliminating interior unknowns on each subdomain is again a DtN map for that subdomain, and iterating between subdomains amounts to iterating on interface data through these maps [9]. Lastly, the DtN map is the central object in a large class of inverse problems. The Calderón problem, the mathematical model behind electrical impedance tomography, asks whether a coefficient (conductivity, or here the wave speed/index of refraction) can be recovered from knowledge of the DtN map alone [3]. Our construction shows that, at least for a fixed known domain D and real k , this map is well-defined and computable through the CFIE machinery above, which is the starting point for asking such inverse questions numerically.

Chapter 4

Numerical Methods

Thus far in these notes we have worked only in the analytic world of integral equations and scattering theory, answering the important questions of existence and uniqueness of solutions. Now we turn towards to the question on every applied mathematician's mind: how do we *actually* solve these problems.

The subject of numerical methods for integral equations and scattering theory could be (and is) the subject of whole textbooks. We do not intend for this to be a comprehensive treatment of numerical methods. Instead we simply wish to highlight the main ideas of the subject, and offer the reader a path forward through this new forest. We begin with kernel approximations, for which we follow [8]. Next we introduce the modern standard of the Nyström method and the difficulties that arise for singular kernels. We end the discussion with corrections schemes for these discretizations.

4.1 Quadrature

We begin this chapter with a brief review of quadrature. Recall a quadrature formula is one such that

$$Q(g) := \int_G w(x)g(x) dx$$

where w is a weight function and g is a continuous function defined on G . We assume that G is compact and Jordan measurable. A quadrature rule is then a discrete approximation to this continuous formula.

$$Q_n(x) := \sum_{j=1}^n \alpha_j g(x_j)$$

Here, x_j are the quadrature nodes where we evaluate the function and $\alpha_j = w(x_j)$ are the weights. For convenience, the order of a quadrature rule n is the number of quadrature nodes we use in the approximation.

Definition 4.1.1. Convergent Quadrature Rule A sequence (Q_n) of quadrature rules is convergent if $Q_n(g) \rightarrow Q(g)$ as $n \rightarrow \infty$ for all $g \in C(G)$.

That is a sequence of quadrature rules is convergent if the linear functionals Q_n converge pointwise to the formula Q . This review will be useful as we consider the numerical solutions to integrals and the Nyström method below.

4.2 Degenerate Kernel Approximations

If we recall our treatment of integral operators with weakly singular kernels in Theorem 1.1.21, we had a kernel we wanted to work with but instead approximated it by a series of more well-behaved kernels. This is the same idea for degenerate kernel approximations.

Let $\langle X, X \rangle$ be a dual system generated by a non-degenerate bilinear form and $A_n : X \rightarrow X$ be a bounded linear operator with

$$A_n \varphi = \sum_{j=1}^n \langle \varphi, b_j \rangle a_j$$

where $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset X$ and the a_i 's are linearly independent.

We call A_n a *degenerate operator* and clearly its action can be expressed as a finite linear combination of basis elements for a space X . In this way, solutions to the second kind equation

$$(I - A_n)\varphi_n = f$$

can be found by solving a finite dimensional linear system.

The justification for this approach is given in the following two theorems.

Theorem 4.2.1. *Let X, Y be Banach spaces and let $A : X \rightarrow Y$ be a bounded linear operator with a bounded inverse $A^{-1} : Y \rightarrow X$. Assume we have a sequence of bounded linear operators $(A_n : X \rightarrow Y)$ that is norm convergent to A . That is $\|A_n - A\| \rightarrow 0$ as $n \rightarrow \infty$. Then for all n with $\|A^{-1}(A_n - A)\| < 1$ the inverse operators A_n^{-1} exist and are bounded by*

$$\|A_n^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|}$$

For solutions of the equations

$$A\varphi = f, \quad \text{and} \quad A_n\varphi_n = f_n$$

we have the error estimate

$$\|\varphi_n - \varphi\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(A_n - A)\|} \{\|(A_n - A)\varphi\| + \|f_n - f\|\}$$

This result tells us that when the integral equation with the convergent operator is uniquely solvable, then the integral equations with the degenerate operators are uniquely solvable.

Theorem 4.2.2. *Assume there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the inverse operators $A_n^{-1} : Y \rightarrow X$ exist and are uniformly bounded. Then the inverse operator $A^{-1} : Y \rightarrow X$ exists and is bounded by*

$$\|A^{-1}\| \leq \frac{\|A_n^{-1}\|}{1 - \|A_n^{-1}(A_n - A)\|}$$

for all n such that $\|A_n^{-1}(A_n - A)\| < 1$.

For solutions of the equations

$$A\varphi = f, \quad \text{and} \quad A_n\varphi_n = f_n$$

we have the error estimate

$$\|\varphi_n - \varphi\| \leq \frac{\|A_n^{-1}\|}{1 - \|A_n^{-1}(A_n - A)\|} \{\|(A_n - A)\varphi_n\| + \|f_n - f\|\}$$

Analogously to the previous theorem, this theorem tells us that if the operators in the sequence of degenerate operators yield unique solutions to $(I - A_n)\varphi_n = f$ then the convergent operator A has a unique solution to $(I - A)\varphi = f$.

Theorem 4.2.3. *Each solution of the equation*

$$\varphi_n - \sum_{j=1}^n \langle \varphi_n, b_j \rangle a_j = f \quad (4.1)$$

has the form

$$\varphi_n = f + \sum_{k=1}^n \gamma_k a_k \quad (4.2)$$

where the coefficients γ_k satisfy the linear system

$$\gamma_k - \sum_{k=1}^n \langle a_k, b_j \rangle \gamma_k = \langle f, b_j \rangle, \quad j = 1, \dots, n \quad (4.3)$$

And then, each solution $\gamma_1, \dots, \gamma_n$ of the linear system (4.3) has a corresponding solution φ_n to (4.1) defined by (4.2).

We can therefore use degenerate approximations of the kernels we have seen for second kind integral equations resulting from the boundary integral formulations of the Laplace and Helmholtz equations. To do so with these integral operators, we can approximate the continuous and weakly singular kernels by a sequence of continuous kernels converging to the kernel in question.

Concretely, if the dual system we are looking at is $\langle C(G), C(G) \rangle$ then the finite dimensional operator $A_n : C(G) \rightarrow C(G)$ has the form

$$(A_n \varphi)(x) = \int_G K_n(x, y) \varphi(y) dy, \quad x \in G$$

where this integral operator has a *degenerate kernel* given by

$$K_n(x, y) = \sum_{j=1}^n a_j(x) b_j(y)$$

In this scenario, as we see from Theorem 4.2.3, the integral equation of the second time becomes

$$\varphi_n(x) - \int_G \sum_{j=1}^n a_j(x) b_j(y) \varphi_n(y) dy = f(x), \quad x \in G$$

and the solution follows from the same theorem. Thus, we can frame the solution $(I - A)\varphi = f$ as the solution of a linear system. We just need to find these degenerate kernels.

4.2.1 Finding Degenerate Kernels

One way we can find these degenerate kernels is through interpolation. Much like the interpolative quadrature methods of classic numerical analysis, we replace the kernel by a piecewise-linear or global periodic interpolation. Then we solve the linear systems with these known interpolants. Explicitly, we interpolate the kernel K with respect to the non-integrating variable

$$K_n(x, y) = \sum_{j=1}^n L_j(x) K(x_j, y)$$

where L_j is the interpolating basis. We have from the analysis of the interpolation method used that $K_n \rightarrow K$ as $n \rightarrow \infty$. The linear system we need to solve then becomes

$$\gamma_j - \sum_{k=1}^n \gamma_k \int_a^b K(x_j, y) L_k(y) dy = \int_a^b K(x_j, y) f(y) dy \quad (4.4)$$

In practice the evaluation of these integrals would require the use of quadrature scheme. With this addition the numerical computation of solutions to integral equations is complete. The error analysis of these degenerate approximations found via interpolation can then be done with the standard tools of approximation theory for interpolation and the quadrature methods used.

Alternatively, we can find the degenerate kernels through expanding the kernel in an orthonormal basis. Let $(\cdot, \cdot)$ be scalar product on $C(G)$ and $\{u_1, u_2, \dots\}$ be a complete orthonormal basis. Then for each fixed $y \in G$ the kernel is expanded with respect to x . Giving us the degenerate kernel

$$K_n(x, y) := \sum_{j=1}^n u_j(x) (K(\cdot, y), u_j)$$

From approximation theory we have that $K_n \rightarrow K$ as $n \rightarrow \infty$. The resulting linear system of this degenerate kernel is then

$$\gamma_j - \sum_{k=1}^n \gamma_k \int_G u_k(y) (K(\cdot, y), u_j) dy = \int_G f(y) (K(\cdot, y), u_j) dy, \quad j = 1, \dots, n$$

Note that the scalar product given is often an integral. So for each matrix coefficient and element of the solution vector in the linear system we need to compute two integrals. This is not very efficient, and is the tradeoff for the relatively easy error analysis we get by using interpolation and basis expansion techniques.

4.3 Nyström Method

The Nyström method is another way to approximate integral equations with continuous or weakly singular kernels. In this scheme, we force the value of the unknown density layer φ to equal the known boundary data f through the boundary integral operator precisely at the quadrature nodes we choose. Whereas before, we were interpolating or expanding with respect to the non-integrating variable, now we discretize this variable so that the integral operator

$$(A\varphi)(x) := \int_G K(x, y) \varphi(y) dy \quad (4.5)$$

is approximated by

$$(A_n \varphi_n)(x) := \sum_{k=1}^n \alpha_k K(x, x_k) \varphi(x_k), \quad x \in G \quad (4.6)$$

The validity of approximating solutions to $(I - A)\varphi = f$ with $(I - A_n)\varphi_n = f_n$ is given by the following theorem.

Theorem 4.3.1. *Let φ_n be a solution of*

$$\varphi_n(x) - \sum_{k=1}^n \alpha_k K(x, x_k) \varphi(x_k) = f(x), \quad x \in G \quad (4.7)$$

then the values $\varphi_j = \varphi_n(x_j)$ $j = 1, \dots, n$ at the quadrature nodes x_j satisfy the linear system

$$\varphi_j - \sum_{k=1}^n \alpha_k K(x_j, x_k) \varphi_k = f(x_j) \quad (4.8)$$

Conversely, let φ_j , $j = 1, \dots, n$ be a solution of the linear system (4.8). Then the function φ_n defined by

$$\varphi_n(x) := f(x) + \sum_{k=1}^n \alpha_k K(x, x_k) \varphi_k, \quad x \in G$$

solves (4.7).

Proof. The forward direction is trivial. For the reverse, evaluate $\varphi_n(x)$ at the quadrature nodes x_j and solve for f . $\square$

The remarkable property of the Nyström method is that the error convergence for the solution to the integral equation is of the same order as the underlying quadrature used to build the discretization [8].

We see one immediate issue with the Nyström method: we will evaluate the kernel at $K(x_j, x_n)$ when $j = n$. We have seen for the Laplace and Helmholtz kernels that they are weakly singular. Thus we have a problem not only when $j = n$ but also for the quadrature nodes near $j = n$. Since here the kernel will be blowing up at a rate that the smooth quadrature will not be able to keep up with. In this neighborhood we see a degradation of the convergence rate. This shows up on the diagonal of the discretized matrix. To combat this, we introduce the idea of correction schemes in the style of Kapur and Rokhlin [5].

The idea of these correction schemes is that we “fix” a small number of quadrature weights near the singularity to get a better convergence rate. For the purposes of this discussion, assume that our scatterer is closed and smooth, so it is parameterized by a global periodic function $f(t)$ and we can use the trapezoidal rule for the quadrature. This means all weights will be the same and we denote them $\alpha = \alpha_1 = \alpha_2 = \dots = \alpha_n$. As stated above, our strategy will be to modify a selection of the quadrature weights. Say we allow ourselves to modify four total weights. Owing to the symmetry of the problem, we really only have two total free variables. Borrowing the notation of [9] we denote the order of the Kapur-Rokhlin correction scheme

as the number of modified weights on each side of the singularity $Q_n^{KR(2)}$. We can write the scheme as

$$Q_n^{KR(2)} = \eta_1 \alpha(f(t_1) + f(t_{n-1})) + \eta_2 \alpha(f(t_2) + f(t_{n-2})) + \underbrace{\sum_{j=1}^n \alpha f(t_j)}_{\text{trap rule}} \quad (4.9)$$

Our job is then to choose these η_1, η_2 so that the singularity blow up can be resolved and we get an improved order of convergence. Kapur and Rokhlin show how to do this in [5] so that error converges like

$$|Q_n^{KR(m)} - Q| = O(h^m), \quad h \rightarrow 0$$

where h is the distance between the equispaced point quadrature nodes of the trapezoid rule. In the case of the log singular kernels of Helmholtz and Laplace, this comes from subtracting the singularity and Taylor expanding. The coefficients of the Taylor expansion give the correction weights. **BP: I need to work this out as an example and add it in.**

This is not the only way to deal with the singularity in the Nyström method and indeed this is an active area of research.

BP: Lots more should be added here. This is just what the presentation on Week 9 covered, but it's a stub. Figures here showing convergence of the correction schemes would do a huge number. As well as those showing the schematics of where the quadrature goes bad.

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